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On commuting unitary operators in spaces with indefinite metric

By M. A. NAĬMARK in Moscow (USSR)

To Professor Béla Szőkefalvi-Nagy on his 50th birthday

Let H be a Hilbert space with the usual inner product $[x, y]$ and with an indefinite inner product (x, y) which, for some complete orthonormal system $\{e_\alpha\}$ in H, is defined by

(1) $$(x, y) = \sum_{\alpha=1}^{\varkappa} \xi_\alpha \bar{\eta}_\alpha - \sum_{\alpha>\varkappa} \xi_\alpha \bar{\eta}_\alpha,$$

where

(2) $$\xi_\alpha = [x, e_\alpha], \quad \eta_\alpha = [y, e_\alpha],$$

$\varkappa$ is a fixed positive integer and $\varkappa < \dim H$. Such a space H will be called a space $\Pi_\varkappa$ with indefinite metric. Another, axiomatic definition of the space $\Pi_\varkappa$ was given by I. S. IOHVIDOV and M. G. KREĬN [1]; we shall follow here the terminology and use the results of this paper.

A linear operator U in $\Pi_\varkappa$ is called unitary if it maps $\Pi_\varkappa$ onto $\Pi_\varkappa$, and preserves the scalar product (x, y), i. e.

$$(Ux, Uy) = (x, y) \text{ for all } x, y \in \Pi_\varkappa.$$

By a theorem of L. S. PONTRYAGIN [2], there exists, for every unitary operator U in $\Pi_\varkappa$, a $\varkappa$-dimensional non-negative subspace, which is invariant with respect to U.[1]) This theorem plays an important role in the study of unitary operators in $\Pi_\varkappa$.

It is therefore natural to expect that the following theorem 1 will be useful for the theory of unitary group representations in $\Pi_\varkappa$, for the theory of rings of operators in $\Pi_\varkappa$, and for other topics[2]).

[1]) L. S. PONTRYAGIN proved his theorem for self-adjoint operators (with respect to (x, y)); using the Cayley transformation one easily sees (*cf.* [1]) that the theorem of L. S. PONTRYAGIN is equivalent to the theorem for unitary operators cited above. Another, simpler proof of the theorem for unitary (and also for more general) operators was given by M. G. KREĬN [2] (see also I. S. IOHVIDOV and M. G. KREĬN [1]; for further generlizations of this theorem see M. BRODSKIĬ [4] and H. LANGER [5], [6]).

[2]) Theorem 1 has been announced by the author in the Note [8] and a proof was there given for $\varkappa = 1$; various applications of the theorem will be treated in further publications. We note that theorem 1 (see also proposition VI. and corollary 2 below) contains the solution for $\Pi_\varkappa$ of a problem posed by PHILLIPS [7].

Theorem 1. *Let $\mathfrak{U}$ be a set of commuting unitary operators in $\Pi_\varkappa$; then there exists in $\Pi_\varkappa$ a $\varkappa$-dimensional non-negative subspace which is invariant with respect to all $U \in \mathfrak{U}$.*

Proof 1. Let $\{e_\alpha\}$ be a complete orthonormal system (with respect to $[x, y]$) in $\Pi_\varkappa$, such that (1) and (2) hold; it follows from (1) and (2) that we also have

(3) $$\xi_\alpha = (x, e_\alpha) \text{ for } \alpha = 1, \dots, \varkappa,$$

(4) $$\xi_\alpha = -(x, e_\alpha) \text{ for } \alpha > \varkappa.$$

Put for any $x \in \Pi_\varkappa$

(5) $$x^+ = \sum_{\alpha=1}^{\varkappa} \xi_\alpha e_\alpha, \quad x^- = \sum_{\alpha>\varkappa} \xi_\alpha e_\alpha;$$

then we have the relation

(6) $$x = \sum_\alpha \xi_\alpha e_\alpha = x^+ + x^-$$

and using (1) and (2) we get:

(7) $$[x, y] = (x^+, x^+) - (x^-, x^-), \quad (x, y) = (x^+, x^+) + (x^-, x^-).$$

We note also that

(8) $$(x^+, x^+) \geqq 0, \quad (x^-, x^-) \leqq 0,$$

and the equality sign holds in (8) only if $x^+ = 0$, or $x^- = 0$, respectively.

Let $X = (x_1, \dots, x_\varkappa)$ be a system of $\varkappa$ vectors $x_1, \dots, x_\varkappa \in \Pi_\varkappa$ satisfying the following conditions:

α) $x_1, \dots, x_\varkappa$ are linearly independent;

β) the $\varkappa$-dimensional subspace $\mathfrak{M}_X$ generated by $x_1, \dots, x_\varkappa$ is non-negative.

I. *The vectors $x_1^+, \dots, x_\varkappa^+$ also are linearly independent.*

In fact, let $\sum_{\alpha=1}^{\varkappa} c_\alpha x_\alpha^+ = 0$ for some numbers c_α. Put $x = \sum_{\alpha=1}^{\varkappa} c_\alpha x_\alpha$; then $x^+ = \sum_{\alpha=1}^{\varkappa} c_\alpha x_\varkappa^+ = 0$. On the other hand, by β), (7), and (8),

$$0 \leqq (x, x) = (x^+, x^+) + (x^-, x^-) = (x^-, x^-) \leqq 0,$$

thus $(x^-, x^-) = 0$, implying $x^- = 0$. Therefore, $x = x^+ + x^- = 0$, i. e. $\sum_{\alpha=1}^{\varkappa} c_\alpha x_\alpha = 0$. By α) this implies $c_1 = c_2 = \dots = c_\varkappa = 0$ concluding the proof of I.

Each vector x_j can be considered as a column of its coordinates $\xi_{\alpha j} = [x_j, e_\alpha]$, thus X will be a matrix $X = \|\xi_{\alpha j}\|$ with $\varkappa$ columns; on the other hand the $x_1^+, \dots, x_\varkappa^+$ define a $\varkappa \times \varkappa$-matrix $X^+ = \|\xi_{\alpha j}\|_{\alpha, j = 1, \dots, \varkappa}$. If X satisfies α) and β), then by I the inverse $(X^+)^{-1}$ exists. A system $X = (x_1, \dots, x_\varkappa)$ satisfying α) and β) will be called *normed*, if $X^+ = 1$, where 1 denotes the $\varkappa \times \varkappa$-identity matrix. If X is not normed, then the matrix $\tilde{X} = X(X^+)^{-1}$ will define a normed system. We denote by K the set of all normed systems satisfying α) and β). Two systems $X = (x_1, \dots, x_\varkappa)$, $X' = (x_1', \dots, x_\varkappa')$ define the same subspace if and only if $X' = XA$, where A is a non-singular $\varkappa \times \varkappa$-matrix.

Particularly, X and $\tilde{X}=X(X^+)^{-1}$ define the same subspace; thus every non-negative $\varkappa$-dimensional subspace is defined by a system $X=(x_1, \ldots, x_\varkappa)\in K$. If two systems $X, X'\in K$ define the same subspace, then $X'=XA$ and hence $X'^+=X^+A$. But $X'^+=X^+=1$ and therefore $A=1$.

In other words:

II. *If $\mathfrak{M}_X$ denotes the subspace defined by a system $X\in K$, then the correspondence $X\to\mathfrak{M}_X$ is a one-to-one mapping of K onto the set of all non-negative $\varkappa$-dimensional subspaces in $\Pi_\varkappa$.*

2. If $X=(x_1, \ldots, x_\varkappa)\in K$, then $(c_1x_1+\ldots+c_\varkappa x_\varkappa, c_1x_1+\ldots+c_\varkappa x_\varkappa)\geqq 0$ holds for any complex $c_1, \ldots, c_\varkappa$. In virtue of (7) this means that

(9) $$[c_1x_1^-+\ldots+c_\varkappa x_\varkappa^-, c_1x_1^-+\ldots+c_\varkappa x_\varkappa^-]\leqq[c_1x_1^++\ldots+c_\varkappa x_\varkappa^+, c_1x_1^++\ldots+c_\varkappa x_\varkappa^+].$$

But condition $X^+=1$ implies that the right hand side of (9) is $\sum_{j=1}^{\varkappa}|c_j|^2$, so that (9) can be written as

(10) $$\sum_{\alpha,\beta=1}^{\varkappa}[x_\alpha^-, x_\beta^-]c_\alpha\bar{c}_\beta\leqq\sum_{j=1}^{\varkappa}|c_j|^2.$$

Conversely, if (10) is satisfied, and if we put $x_j=e_j+x_j^-$, $j=1, \ldots, \varkappa$, we get a system $X=(x_1, \ldots, x_\varkappa)\in K$. If

$$c_{\alpha'}=\begin{cases}1 & \text{for } \alpha'=\alpha\\ 0 & \text{for } \alpha'\neq\alpha\end{cases}$$

then (10) takes the form

(11) $$[x_\alpha^-, x_\alpha^-]\leqq 1$$

and hence

(12) $$\sum_{\alpha=1}^{\varkappa}[x_\alpha^-, x_\alpha^-]\leqq\varkappa.$$

By (5) each x_α^- can be represented in the form

(13) $$x_\alpha^-=\sum_{\beta>\varkappa}\xi_{\beta\alpha}e_\beta \quad\text{where}\quad \xi_{\beta\alpha}=[x_\alpha^-, e_\beta];$$

thus (12) can be written as

(14) $$\sum_{\alpha=1}^{\varkappa}\sum_{\beta>\varkappa}|\xi_{\beta\alpha}|^2\leqq\varkappa.$$

Denote by $\mathfrak{H}$ the Hilbert space of all sequences $\xi=\{\xi_{\beta\alpha}; \alpha=1, \ldots, \varkappa; \beta>\varkappa\}$ with the norm

$$\|\xi\|=\left(\sum_{\alpha=1}^{\varkappa}\sum_{\beta>\varkappa}|\xi_{\beta\alpha}|^2\right)^{\frac{1}{2}}<\infty$$

and let Q be the ball $\|\xi\|^2\leqq\varkappa$ in $\mathfrak{H}$. Then (13) and (14) mean:

III. *The correspondence $X\to\xi=\{\xi_{\beta\alpha}; \alpha=1, \ldots, \varkappa; \beta>\varkappa\}$ is a one-to-one mapping of K onto a set $Q_1\subset Q$.*

The ball Q is known to be bicompact in the weak topology of $\mathfrak{H}$. On the other hand Q_1 is closed[3]) and hence also bicompact. In fact by (10) and (13) $\xi \in Q_1$ if and only if

$$\sum_{\gamma>\varkappa}\left|\sum_{\alpha=1}^{\varkappa}\xi_{\gamma\alpha}c_\alpha\right|^2 \leqq \sum_{j=1}^{\varkappa}|c_j|^2. \tag{15}$$

Let Γ be a finite set $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ and G be the family of all finite sets Γ (with any number of elements). Denote by $Q(\Gamma, c_1, \ldots, c_\varkappa)$ the set of all $\xi \in \mathfrak{H}$ satisfying the inequality

$$\sum_{\alpha,\beta=1}^{\varkappa}\sum_{\gamma\in\Gamma}\xi_{\gamma\alpha}\bar{\xi}_{\gamma\beta}c_\alpha\bar{c}_\beta \leqq \sum_{j=1}^{\varkappa}|c_j|^2 \tag{16}$$

for fixed $c_1, \ldots, c_\varkappa$ and Γ, and let $Q(c_1, \ldots, c_\varkappa)$ denote the set of all $\xi \in \mathfrak{H}$ satisfying (15) for fixed $c_1, \ldots, c_\varkappa$. Each $\xi_{\gamma\alpha}$ is a continuous function of ξ, hence the left hand side of (16) also is a continuous function. Therefore, $Q(\Gamma, c_1, \ldots, c_\varkappa)$ is closed. But

$$Q(c_1, \ldots, c_\varkappa) = \bigcap_{\Gamma\in G} Q(\Gamma, c_1, \ldots, c_\varkappa)$$

and

$$Q_1 = \bigcap_{c_1, \ldots, c_\varkappa} Q(c_1, \ldots, c_\varkappa),$$

where the last intersection is taken over all systems $c_1, \ldots, c_\varkappa$ of complex numbers. Thus $Q(c_1, \ldots, c_\varkappa)$ and Q_1 are also closed and Q_1 is a bicompact set.

Now we show that Q_1 is a convex set. Denote by $l^2_\varkappa$ the $\varkappa$-dimensional Hilbert space of all $c=(c_1, \ldots, c_\varkappa)$ with the inner product $(c, c') = \sum_{j=1}^{\varkappa} c_j\bar{c}'_j$ and let l^2 be the Hilbert space of all sequences $\eta = \{\eta_\gamma, \gamma > \varkappa\}$ satisfying $\sum_{\gamma>\varkappa}|\eta_\gamma| < \infty$ with the inner product

$$(\eta, \eta') = \sum_{\gamma>\varkappa}\eta_\gamma\bar{\eta}'_\gamma.$$

Then in virtue of (15) Q_1 can be regarded as the set of all bounded operators

$$\eta_\gamma = \sum_{\alpha=1}^{\varkappa}\xi_{\gamma\alpha}c_\alpha$$

from $l^2_\varkappa$ to l^2 with norm $\leqq 1$. As the last set is convex, Q_1 is also convex.

3. Let $X=(x_1, \ldots, x_\varkappa)$ be a system satisfying α) and β) (p. 178), and let U be a unitary operator in $\Pi_\varkappa$. Then the system $Y=(Ux_1, \ldots, Ux_\varkappa)$ satisfies also α) and β).

In fact, since U is unitary and $x_1, \ldots, x_\varkappa$ are linearly independent, $Ux_1, \ldots, Ux_\varkappa$ are also linearly independent. Further, using β) for $x_1, \ldots, x_\varkappa$ we have

$$\left(\sum_{\alpha=1}^{\varkappa}c_\alpha Ux_\alpha, \sum_{\alpha=1}^{\varkappa}c_\alpha Ux_\alpha\right) = \left(U\sum_{\alpha=1}^{\varkappa}c_\alpha x_\alpha, U\sum_{\alpha=1}^{\varkappa}c_\alpha x_\alpha\right) = \left(\sum_{\alpha=1}^{\varkappa}c_\alpha x_\alpha, \sum_{\alpha=1}^{\varkappa}c_\alpha x_\alpha\right) \geqq 0.$$

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[3]) In the following all topological notions in $\mathfrak{H}$ will be considered in the weak topology of $\mathfrak{H}$.

Particularly, U transforms every system $X \in K$ (*cf.* p. 178) into a system Y satisfying $\alpha)$ and $\beta)$, hence by I the matrix $(Y^+)^{-1}$ exists. We denote by V_U the (non-linear) operator defined by

$$V_U X = Y(Y^+)^{-1} \quad \text{where} \quad Y = (Ux_1, \ldots, Ux_\varkappa). \tag{17}$$

As $V_U X \in K$, the operator V_U transforms K into itself. By virtue of III V_U can also be considered as an operator V_U transforming Q_1 into itself. This operator V_U is continuous in Q_1. In fact, let Π^+ (and Π^-) denote the set of all $x \in \Pi_\varkappa$ for which $x^- = 0$ (resp. $x^+ = 0$); then in virtue of (6) and (7)

$$\Pi_\varkappa = \Pi^+ \oplus \Pi^- \tag{18}$$

where Π^+ and Π^- are orthogonal with respect to (x, y) and also with respect to $[x, y]$. According to the decomposition (18) U can be given by a matrix

$$U \sim \begin{Vmatrix} A & B \\ C & D \end{Vmatrix} \tag{19}$$

where A, B, C, D are bounded operators; A is an operator in Π^+, D is an operator in Π^-, B is an operator from Π^- into Π^+ and C is an operator from Π^+ into Π^-. If we use the orthonormal system $\{e_\alpha\}$ and the decompositions (5), we see that Π^+ and Π^- coincide with the spaces $l^2_\varkappa$ and l^2, and A, B, C, D are represented by matrices. Moreover, the systems $X \in K$ are represented by matrices

$$X = \begin{Vmatrix} 1 \\ \xi \end{Vmatrix}$$

where 1 is the $\varkappa \times \varkappa$-identity matrix and in virtue of (19) $Y = (Ux_1, \ldots, Ux_\varkappa)$ means that

$$Y = \begin{Vmatrix} A + B\xi \\ C + D\xi \end{Vmatrix}.$$

Thus $Y^+ = A + B\xi$ and $V_U \xi = (C + D\xi)(A + B\xi)^{-1}$. As A, B, C, D are bounded, the functions $\xi \to C + D\xi$ and $\xi \to A + B\xi$ are continuous. Moreover $A + B\xi$ is a $\varkappa \times \varkappa$-matrix and $(A + B\xi)^{-1}$ exists; hence the function $\xi \to (A + B\xi)^{-1}$ is also continuous. Thus the function $\xi \to V_U \xi = (C + D\xi)(A + B\xi)^{-1}$ is also continuous. Therefore V_U is a continuous transformation into itself of the convex bicompact set Q_1 and hence V_U has a fixpoint in Q_1. Let ξ be such a fixpoint, i. e.

$$V_U \xi = \xi.$$

In virtue of (17) and III this means that

$$Y(Y^+)^{-1} = X \quad \text{hence} \quad Y = XY^+$$

i. e. the systems $Y = (Ux_1, \ldots, Ux_\varkappa)$ and $X = (x_1, \ldots, x_\varkappa)$ define the same subspace $\mathfrak{M}_X$; this means that $\mathfrak{M}_X$ is invariant with respect to U. So we have proved the *existence* of a non-negative $\varkappa$-dimensional subspace, which is invariant with respect to U.[4])

[4]) The argument in sections 2 and 3 is a slight modification of the proof of theorem 3. 1 in [1].

4. We have proved in section 3, that every fixpoint ξ of V_U in Q_1 defines a non-negative $\varkappa$-dimensional subspace $\mathfrak{M}_X$ in $\Pi_\varkappa$, which is invariant with respect to U.

Conversely, let $\mathfrak{M}$ be any non-negative $\varkappa$-dimensional subspace in $\Pi_\varkappa$ which is invariant with respect to U. In virtue of II $\mathfrak{M}=\mathfrak{M}_X$ for some uniquely defined $X\in K$ and the invariance of $\mathfrak{M}_X$ means that $X=(x_1, \ldots, x_\varkappa)$ and $Y=(Ux_1, \ldots, Ux_\varkappa)$ define the same subspace, i. e.

$$Y=XA \tag{20}$$

where A is a $\varkappa\times\varkappa$-matrix. As $X^+=1$ this implies $Y^+=X^+A=A$ and hence

$$Y=XY^+, \quad Y(Y^+)^- =X.$$

But this means that $V_U\xi=\xi$, where ξ is defined by

$$X=\begin{Vmatrix} 1 \\ \xi \end{Vmatrix}$$

i. e. ξ is a fixpoint of V_U. In other words:

IV. *The mappings $\mathfrak{M}_X\leftrightarrow X\leftrightarrow\xi$ in propositions* II *and* III *define a one-to-one correspondence $\mathfrak{M}_X\leftrightarrow\xi$ between all non-negative $\varkappa$-dimensional subspaces $\mathfrak{M}$, which are invariant with respect to U and all fixpoints ξ in Q_1 of V_U.*

5. Denote by Q_U the set of all fixpoints of V_U in Q. As V_U is continuous Q_U is closed. In virtue of IV our theorem will be proved if we show that the intersection of all Q_U $(U\in\mathfrak{U})$ is not void. But Q_1 being bicompact it suffices to prove that the intersection of every finite system $Q_{U_1}, \ldots, Q_{U_n}$ $(U_j\in\mathfrak{U})$ is not void. In virtue of IV this means that for every finite system $U_1, \ldots, U_n$ of commuting unitary operators there exists a non negative $\varkappa$-dimensional subspace, which is invariant with respect to every U_j $(j=1, \ldots, n)$.

We prove first the following weaker assertion:

V. *For any commuting unitary operators $U_1, \ldots, U_n$ in $\Pi_\varkappa$ a non-negative subspace $\mathfrak{N}\neq(0)$ (not necessarily $\varkappa$-dimensional) exists, which is invariant with respect to $U_1, \ldots, U_n$.*

We prove this proposition by induction with respect to n. For $n=1$ the assertion V follows from the assertion proved in section 3. We suppose that the assertion is true for some n and prove it to be also true for $n+1$.

Let $U_1, \ldots, U_n, U_{n+1}$ be commuting unitary operators in $\Pi_\varkappa$. By our assumption a non-negative subspace $\mathfrak{N}\neq(0)$ exists, which is invariant with respect to $U_1, \ldots, U_n$; by Lemma 1. 2 in [1] $\mathfrak{N}$ is finite dimensional and $\dim\mathfrak{N}\leqq\varkappa$. The restrictions of $U_1, \ldots, U_n$ to $\mathfrak{N}$ are commuting linear operators in the finite dimensional space $\mathfrak{N}$. Hence they have a common eigenvector, say $x_0\neq 0$ in $\mathfrak{N}$. Thus

$$U_jx_0=\lambda_jx_0 \quad \text{for } j=1, \ldots, n \tag{21}$$

where λ_j is the eigenvalue of U_j corresponding to x_0; as $x_0\in\mathfrak{N}$,

$$(x_0, x_0)\geqq 0. \tag{22}$$

Denote by $\mathfrak{N}'$ the set of all vectors $x \in \Pi_\varkappa$ satisfying

(23) $$U_j x = \lambda_j x \quad \text{for } j = 1, \ldots, n.$$

Then by (24) we have $x_0 \in \mathfrak{N}'$.

Moreover we have

(24) $$U_{n+1}\mathfrak{N}' = \mathfrak{N}'.$$

In fact, if $x \in \mathfrak{N}'$, i. e. (23) holds, then $U_{n+1}U_j x = \lambda_j U_{n+1} x$, i. e. $U_j U_{n+1} x = \lambda_j U_{n+1} x$. This means that $U_{n+1}\mathfrak{N}' \subset \mathfrak{N}'$. Replacing in this argument U_{n+1} by U_{n+1}^{-1} we also get $U_{n+1}^{-1}\mathfrak{N}' \subset \mathfrak{N}'$, hence $\mathfrak{N}' \subset U_{n+1}\mathfrak{N}'$ concluding the proof of (24). As $\mathfrak{N}'$ contains the non-negative vector x_0', theorem 4. 4 in [1] can be applied to $\mathfrak{N}'$ and U_{n+1}. Thus $\mathfrak{N}'$ contains a non-negative subspace $\mathfrak{N}_0' \neq (0)$ which is invariant with respect to U_{n+1}. By (23) $\mathfrak{N}_0'$ is also invariant with respect to $U_1, \ldots, U_n$. This concludes the proof of proposition V.

We prove now the following proposition:

VI. *Let* $\mathfrak{N} \neq (0)$ *be a non-negative subspace, which is invariant with respect to* $U_1, \ldots, U_n$. *If* $\dim \mathfrak{N} < \varkappa$ *then a non-negative subspace* $\mathfrak{N}_1 \supset \mathfrak{N}$ *exists,* $\mathfrak{N}_1 \neq \mathfrak{N}$, *which is also invariant with respect to* $U_1, \ldots, U_n$.

If proposition VI is proved, then applying it first to $\mathfrak{N}$, then to $\mathfrak{N}_1$, and so on, we get after a finite number of steps a $\varkappa$-dimensional non-negative subspace $\mathfrak{M}$ which is invariant with respect to $U_1, \ldots, U_n$ and this concludes the proof of Theorem 1. So, we turn to the proof of proposition VI.

Let $\dim \mathfrak{N} = \varkappa_0 < \varkappa$. Only the following cases are possible:

a) $\mathfrak{N}$ is positive. Then $\mathfrak{N}^\perp$ is a space $\Pi_{\varkappa-\varkappa_0}$ and $\mathfrak{N}^\perp$ is also invariant[5]) with respect to $U_1, \ldots, U_n$. Applying proposition V to the restrictions of $U_1, \ldots, U_n$ to $\mathfrak{N}^\perp$ we get a non-negative subspace $\mathfrak{N}' \subset \mathfrak{N}^\perp$, $\mathfrak{N}' \neq (0)$, which is invariant with respect to $U_1, \ldots, U_n$. Put $\mathfrak{N}_1 = \mathfrak{N} \oplus \mathfrak{N}'$. Then $\mathfrak{N} \subset \mathfrak{N}_1$, $\mathfrak{N} \neq \mathfrak{N}_1$, $\mathfrak{N}_1$ is non-negative and invariant with respect to $U_1, \ldots, U_n$.

b) $\mathfrak{N}$ is a nullspace. Let G be a subspace in $\Pi_\varkappa$ skewly related to $\mathfrak{N}$ (*cf.* [1], definition 4. 1); put $F = \mathfrak{N} \dotplus G$ and $H = \mathfrak{N}^\perp$. Then F is a $2\varkappa_0$-dimensional space $\Pi_{\varkappa_0}$, hence $F^\perp$ is a space $\Pi_{\varkappa-\varkappa_0}$. Thus

$$\Pi_\varkappa = (\mathfrak{N} \dotplus G) \oplus \Pi_{\varkappa-\varkappa_0}.$$

Using the argument in the proof of Lemma 4. 1 in [1] we get

(25) $$H = \mathfrak{N}^\perp = \mathfrak{N} \oplus \Pi_{\varkappa-\varkappa_0}.$$

As $\mathfrak{N} \perp H$, relation (25) shows that the factor-space $\tilde{H} = H/\mathfrak{N}$ is isomorphic to $\Pi_{\varkappa-\varkappa_0}$ and hence is also a space $\Pi_{\varkappa-\varkappa_0}$. On the other hand, $\mathfrak{N}$ being invariant with respect to the unitary operators $U_1, \ldots, U_n$ the subspace $H = \mathfrak{N}^\perp$ has the same property (see the footnote [5]); hence the U_j $(j = 1, \ldots, n)$ induce commuting unitary operators $\tilde{U}_j$ $(j = 1, \ldots, n)$ in $\tilde{H} = \Pi_{\varkappa-\varkappa_0}$. In virtue of V, there exists a non-negative

[5]) In fact as $\mathfrak{N}$ is finite dimensional, and U_j are unitary, we have $U_j\mathfrak{N} = \mathfrak{N}$ and therefore for $x \in \mathfrak{N}^\perp$, $y \in \mathfrak{N}$ we get

$$(U_j x, y) = (x, U_j^{-1} y) = 0$$

in virtue of $U_j^{-1} y \in \mathfrak{N}$. This shows, that $U_j x \in \mathfrak{N}^\perp$.

subspace $\tilde{\mathfrak{N}} \neq (0)$, $\tilde{\mathfrak{N}} \subset H$, which is invariant with respect to $\tilde{U}_1, \ldots, \tilde{U}_n$. Let f be the natural mapping of H onto $\tilde{H}$; put $\mathfrak{N}_1 = f^{-1}(\tilde{\mathfrak{N}})$. Then $\mathfrak{N} \subset \mathfrak{N}_1$, $\mathfrak{N} \neq \mathfrak{N}_1$, $\mathfrak{N}_1$ is non-negative and invariant with respect to $U_1, \ldots, U_n$.

c) $\mathfrak{N}$ is not a nullspace, but it contains nullvectors. By the Cauchy—Bunyakovsky inequality, valid in $\mathfrak{N}$, each such nullvector is isotropic for $\mathfrak{N}$, hence the set of all nullvectors in $\mathfrak{N}$ coincides with the isotropic subspace of $\mathfrak{N}$, which we denote by $\mathfrak{N}'$. By our assumption $(0) \neq \mathfrak{N}' \subset \mathfrak{N}$, $\mathfrak{N}' \neq \mathfrak{N}$, and therefore $0 < \varkappa' < \varkappa_0$, where $\varkappa' = \dim \mathfrak{N}'$. Let G be a subspace in $\Pi_\varkappa$, which is skewly related to $\mathfrak{N}'$. Put

$$\mathfrak{N}'' = \{x : x \in \mathfrak{N}, x \perp G\} = \mathfrak{N} \cap G^\perp.$$

Then

$$\mathfrak{N} = \mathfrak{N}' \oplus \mathfrak{N}''. \tag{26}$$

In fact, $\mathfrak{N}', \mathfrak{N}'' \subset \mathfrak{N}$ and $\mathfrak{N}' \perp \mathfrak{N}$; hence $\mathfrak{N}' \oplus \mathfrak{N}'' \subset \mathfrak{N}$ and we have to prove the opposite inclusion $\mathfrak{N}' \oplus \mathfrak{N}'' \supset \mathfrak{N}$. By Lemma 4. 1 in [1] we have

$$\Pi_\varkappa = \mathfrak{N}' \dotplus G^\perp$$

so that any $x \in \Pi_\varkappa$ can be uniquely represented in the form $x = y + z$, where $y \in \mathfrak{N}'$, $z \in G^\perp$. If now $x \in \mathfrak{N}$, then $z = x - y \in \mathfrak{N}$ thus $z \in \mathfrak{N} \cap G^\perp = \mathfrak{N}''$ and $x = y + z \in \mathfrak{N}' \oplus \mathfrak{N}''$ concluding the proof of (26).

The subspace $\mathfrak{N}''$ is positive. In fact, if $x \in \mathfrak{N}''$ and $(x, x) = 0$ then x is an isotropic vector for $\mathfrak{N}$, hence $x \in \mathfrak{N}'$. Thus x is an element of $\mathfrak{N}'$, which is orthogonal to G; by the definition of G this is impossible if $x \neq 0$. The last argument show that $\mathfrak{N}' \cap \mathfrak{N}'' = (0)$, so that in virtue of (26)

$$\varkappa_0 = \varkappa' + \varkappa'', \quad \text{where, } \varkappa'' = \dim \mathfrak{N}''. \tag{27}$$

Now put (*cf.* (26))

$$F = \mathfrak{N} \dotplus G = (\mathfrak{N}'' \oplus \mathfrak{N}') \dotplus G = \mathfrak{N}'' \oplus (\mathfrak{N}' \dotplus G) \tag{28}$$

and

$$H = \mathfrak{N}^\perp, \quad H' = F^\perp. \tag{29}$$

As $\mathfrak{N}''$ is a positive $\varkappa''$-dimensional subspace and $\mathfrak{N}' \dotplus G$ is a $2\varkappa'$-dimensional space $\Pi_{\varkappa'}$ equality (28) implies that F is a $2\varkappa' + \varkappa''$-dimensional space $\Pi_{\varkappa' + \varkappa''} = \Pi_{\varkappa_0}$. Therefore H' is a space $\Pi_{\varkappa - \varkappa_0}$. Moreover,

$$H = H' \oplus \mathfrak{N}'. \tag{30}$$

In fact, as $F \supset \mathfrak{N}$, we have $H' = F^\perp \subset \mathfrak{N}^\perp = H$ and also $\mathfrak{N}' \subset H$, hence $H' \oplus \mathfrak{N}' \subset H$. So we have to prove the opposite relation $H \subset H' \oplus \mathfrak{N}'$, or what is the same $\mathfrak{N} = H^\perp \supset \supset (H' \oplus \mathfrak{N}')^\perp$. Let $x \in (H' \oplus \mathfrak{N}')^\perp$. Then $x \in H'^\perp = F$ and by (28) we have $x = y + z$, where $y \in \mathfrak{N}$, $z \in G$. On the other hand, we have $\mathfrak{N}' \perp \mathfrak{N}$, hence $y \perp \mathfrak{N}'$ and therefore $z = x - y \perp \mathfrak{N}'$. As G and $\mathfrak{N}'$ are skewly related, this implies $z = 0$; then $x = y \in \mathfrak{N}$ concluding the proof of (30).

The subspaces $\mathfrak{N}$ and $H = \mathfrak{N}^\perp$ are invariant with respect to $U_1, \ldots, U_n$. Hence $\mathfrak{N}' = \mathfrak{N}^\perp \cap \mathfrak{N} = H \cap \mathfrak{N}$ is also invariant with respect to $U_1, \ldots, U_n$ and therefore the $U_j (j = 1, \ldots, n)$ induce commuting unitary operators $\tilde{U}_j (j = 1, \ldots, n)$ in the factor-space

$$\tilde{H} = H/\mathfrak{N}'. \tag{31}$$

But in virtue of (30) $\tilde{H}$ is isomorphic to H' and therefore is a space $\Pi_{\varkappa-\varkappa_0}$.

By proposition V there exists a non-negative subspace $\tilde{\mathfrak{N}} \subset \tilde{H}$, $\tilde{\mathfrak{N}} \neq (0)$, which is invariant with respect to $\tilde{U}_1, \ldots, \tilde{U}_n$. Let f be the natural mapping of H onto $\tilde{H}$; put $\mathfrak{N}^* = f^{-1}(\tilde{\mathfrak{N}})$. Then $\mathfrak{N}^*$ is a non-negative subspace, which is invariant with respect to $U_1, \ldots, U_n$, $\mathfrak{N}^* \supset \mathfrak{N}'$, $\mathfrak{N}^* \neq \mathfrak{N}'$, and $\mathfrak{N}^* \subset H$; hence $\mathfrak{N}^* \perp \mathfrak{N}$. Put

$$\mathfrak{N}_1 = \mathfrak{N} \oplus \mathfrak{N}^*;$$

then $\mathfrak{N}_1$ is a non-negative subspace, which is invariant with respect to $U_1, \ldots, U_n$ and it remains to show that $\dim \mathfrak{N}_1 > \dim \mathfrak{N}$. To this end we note that

$$\mathfrak{N}' \subset \mathfrak{N} \cap \mathfrak{N}^* \subset \mathfrak{N} \cap H = \mathfrak{N}',$$

hence $\mathfrak{N} \cap \mathfrak{N}^* = \mathfrak{N}'$ and therefore

$$\dim \mathfrak{N}_1 = \dim \mathfrak{N} + \dim \mathfrak{N}^* - \dim \mathfrak{N}' > \dim \mathfrak{N},$$

concluding the proof of proposition VI and theorem 1.

Corollary 1. *For every family $\mathcal{H}$ of commuting bounded Hermitian operators in $\Pi_\varkappa$ there exists a $\varkappa$-dimensional non-negative subspace, which is invariant with respect to all operators of $\mathcal{H}$.*

Proof. Put for real t and $H \in \mathcal{H}$

$$U_t = e^{itH} = 1 + \frac{t}{1!}(iH) + \frac{t^2}{2!}(iH)^2 + \ldots.$$

Then the U_t form a commuting set of unitary operators in $\Pi_\varkappa$. By Theorem 1, there exsits a $\varkappa$-dimensional non-negative subspace $\mathfrak{M}$, which is invariant with respect to all e^{itH}, $H \in \mathcal{H}$, $t \in (-\infty, \infty)$. In virtue of the relation

$$\left\| \frac{1}{it}(e^{itH} - 1) - H \right\| \to 0 \quad \text{for} \quad t \to 0,$$

$\mathfrak{M}$ is also an invariant subspace for all $H \in \mathcal{H}$.

Corollary 2. *Let R be a commutative algebra of bounded operators in $\Pi_\varkappa$, satisfying the condition: $A \in R$ implies $A^* \in R$ where A^* is the adjoint operator with respect to (x, y) (i. e. $(Ax, y) = (x, A^*y)$ for all $x, y \in \Pi_\varkappa$). Then a non-negative $\varkappa$-dimensional subspace exists which is invariant with respect to all $A \in R$.*

Proof. Ler $\mathcal{H}$ be the set of all Hermitian operators from R. Then $\mathcal{H}$ satisfies the conditions of Corollary 1. Hence a $\varkappa$-dimensional non-negative subspace $\mathfrak{M}$ exists, which is invariant with respect to all $H \in \mathcal{H}$. If now $A \in R$, then also $A^* \in R$ and we have $A = H_1 + iH_2$, where $H_1 = \frac{1}{2}(A + A^*)$, $H_2 = \frac{1}{2i}(A - A^*)$. Thus

H_1, H_2 are Hermitian, $H_1, H_2 \in R$ and therefore $H_1, H_2 \in \mathfrak{H}$. As $\mathfrak{M}$ is invariant with respect to $H_1, H_2 \in \mathfrak{H}$ it is also invariant with respect to A.

The following Theorem 2 generalizes Corollary 2; assertion 2) of this theorem can be considered as an infinite dimensional generalization of the Lie theorem for solvable Lie algebras.

Theorem 2. *Let $X_0, X_1, X_2, \ldots, X_m$ be sets of linear bounded operators in $\Pi_\varkappa$, and $H_0, H_1, \ldots, H_{m-1}$ bounded Hermitian operators such that* a) $X_0 \supset X_1 \supset \ldots \supset X_m$; b) *$X_\nu$ is generated by H_ν and $X_{\nu+1}$ for $\nu = 0, 1, \ldots, m-1$;* c) *$[H_\nu, A] = H_\nu A - AH_\nu \in X_{\nu+1}$ for every $A \in X_{\nu+1}$;* d) *X_m is commutative and $A \in X_m$ implies $A^* \in X_m$.*

Then: 1) *there exists a non-negative $\varkappa$-dimensional subspace in $\Pi_\varkappa$ which is invariant with respect to all operators from X_0;* 2) *there exists a non-negative vector $x_0 \in \Pi_\varkappa$, $x_0 \neq 0$ which is a common eigenvector for all operators from X_0.*

Proof. We prove first by induction, that $A \in X_\nu$ implies $A^* \in X_\nu$ for $\nu = 0, 1, \ldots, m-1$. For $\nu = m$ this assertion follows from the condition d) of the theorem. Now we suppose the assertion is true for some $\nu+1$ and prove it to be true for ν. Let $A \in X_\nu$; then by condition b) $A = \alpha H_\nu + A_1$, where $A_1 \in X_{\nu+1}$, hence $A_1^* \in X_{\nu+1}$. But then $A^* = \bar{\alpha} H_\nu + A_1^* \in X_\nu$ and the assertion is proved for ν. Denote by $\mathfrak{H}_\nu$ the set of all Hermitian operators from X_ν. Using the assertion proved and applying the same argument as in the proof of Corollary 2 we see that every $A \in X_\nu$ has the form

$$A = H_1 + iH_2 \quad (H_1, H_2 \in \mathfrak{H}_\nu). \tag{32}$$

Now we prove assertion 2) by induction. For X_m the assertion follows from Corollary 1. In fact, by Corollary 1 a non-negative k-dimensional subspace $\mathfrak{M}$ exists, which is invariant with respect to all $H \in \mathfrak{H}_m$; in virtue of (32) $\mathfrak{M}$ is also invariant with respect to all $A \in X_m$. As $\mathfrak{M}$ is finite dimensional and invariant with respect to the commuting family X_m, there exists a vector $x_0 \in \mathfrak{M}$, $x_0 \neq 0$, which is a common eigenvector for all $A \in X_m$.

Now we suppose that assertion 2) holds for some $X_{\nu+1}$ and then prove it to hold for X_ν. By our assumption, there exists a non-negative vector $x_0 \neq 0$, $x_0 \in \Pi_\varkappa$, which is a common eigenvector for all $A \in X_{\nu+1}$, so that

$$Ax_0 = \lambda(A) x_0 \text{ for all } A \in X_{\nu+1}, \tag{33}$$

where $\lambda(A)$ is a complex-valued linear function on $X_{\nu+1}$. Put

$$H_\nu^p x_0 = x_p \qquad (p = 0, 1, 2, 3, \ldots) \tag{34}$$

and

$$[A, H_\nu] = A^{(1)}, \qquad [A^{(p)}, H_\nu] = A^{(p+1)} \qquad (p = 0, 1, 2, \ldots) \tag{35}$$

where by definition $A^{(0)} = A$.

Then in virtue of condition c) of the theorem

$$A^{(p)} \in X_{\nu+1} \quad \text{for all} \quad A \in X_{\nu+1} \quad \text{and} \quad p = 1, 2, 3, \ldots,$$

hence by (33) and (34)

$$Ax_1 = AH_\nu x_0 = [A, H_\nu]x_0 + H_\nu Ax_0 = A^{(1)}x_0 + \lambda(A)H_\nu x_0 = \\ = \lambda(A^{(1)})x_0 + \lambda(A)x_1.$$

Repeating the same argument we easily obtain by induction, that

(36) $$Ax_p = \lambda(A)x_p + p\lambda(A^{(1)})x_{p-1} + C_p^2\lambda(A^{(2)})x_{p-2} + \ldots \\ \ldots + C_p^q\lambda(A^{(q)})x_{p-q} + \ldots + \lambda(A^{(p)})x_0$$

holds for all $A \in X_{\nu+1}$ and all $p = 1, 2, 3, \ldots$.

We show that in fact $\lambda(A^{(1)}) \equiv 0$ and hence also $\lambda(A^{(p)}) \equiv 0$ for all $p = 1, 2, 3, \ldots$ and $A \in X_{\nu+1}$. Suppose the contrary; let $\lambda(A^{(1)}) \not\equiv 0$; then in virtue of (32) also $\lambda(A^{(1)}) \not\equiv 0$ on $\mathcal{H}_{\nu+1}$. Only the following cases can occur:

Case α): $\lambda(A)$ is not real for some $A = A_0 \in \mathcal{H}_{\nu+1}$. Then $(x_0, x_0) = 0$. We show by induction, that

(37) $$(x_q, x_r) = 0$$

holds for all $q, r = 1, 2, \ldots$. First we remark, that

(38) $$(x_q, x_r) = (H_\nu^q x_0, H_\nu^2 x_0) = (H_\nu^{q+2}x_0, x_0),$$

so that (x_q, x_r) depends only on $q+r$.

We have seen that $(x_0, x_0) = 0$, hence our assertion holds for $q+r = 0$. We suppose it is true for $q+r<p$ and prove it to be true for $q+r = p$. To this end we take the inner product of both sides of (36) with x_0. Then by our inductive assumption we get

$$(Ax_p, x_0) = \lambda(A)(x_p, x_0)$$

and on the other hand for $A \in \mathcal{H}_{\nu+1}$ we have

$$(Ax_p, x_0) = (x_p, Ax_0) = (x_p, \lambda(A)x_0) = \overline{\lambda(A)}(x_p, x_0);$$

thus

$$[\lambda(A) - \overline{\lambda(A)}](x_p, x_0) = 0.$$

But $\lambda(A_0) - \overline{\lambda(A_0)} \neq 0$, hence $(x_p, x_0) = 0$ concluding the proof of (37). Denote by $\mathfrak{M}$ the closed subspace generated by all x_p ($p = 0, 1, 2, \ldots$). By (34) $\mathfrak{M}$ is invariant with respect to H_ν. In virtue of (37) $\mathfrak{M}$ is a nullspace in $\Pi_\varkappa$ and hence dim $\mathfrak{M} \leq \varkappa$, $\mathfrak{M}$ is finite-dimensional. Relations (36) show, that $\mathfrak{M}$ is also invariant with respect to A. Let $\tilde{A}, \tilde{H}_\nu$ be the restrictions of A and H_ν to $\mathfrak{M}$; then (36) holds also for these $\tilde{A}$ and $\tilde{H}_\nu$. Put in (36) $\tilde{A}^{(1)}$ instead of $\tilde{A}$; then we obtain

$$\tilde{A}^{(1)}x_p = \lambda(A^{(1)})x_p + p\lambda(A^{(2)})x_{p-1} + \ldots + \lambda(A^{(p+1)})x_0 \qquad (p = 0, 1, 2, \ldots).$$

These equalities show that by our assumption

$$\mathrm{Tr}(\tilde{A}^{(1)}) = \lambda(A^{(1)}) \dim \mathfrak{M} \neq 0 \quad \text{for some} \quad A \in \mathcal{H}_{\nu+1}$$

where $\mathrm{Tr}(\tilde{A})$ denotes the trace of $\tilde{A}$.

On the other hand we have

$$\mathrm{Tr}(\tilde{A}^{(1)}) = \mathrm{Tr}(\tilde{A}\tilde{H}_\nu - \tilde{H}_\nu \tilde{A}) = \mathrm{Tr}(\tilde{A}\tilde{H}_\nu) - \mathrm{Tr}(\tilde{H}_\nu \tilde{A})$$

and we get a contradiction which shows that $\lambda(A^{(1)}) \not\equiv 0$ is impossible in case α).

Case β): $\lambda(A)$ is real for all $A \in \mathcal{H}_{\nu+1}$. For $A \in \mathcal{H}_{\nu+1}$ we have

$$A^{(1)*} = (AH_\nu - H_\nu A)^* = H_\nu A - AH_\nu = -A^{(1)},$$

thus $A^{(1)}$ has the form

$$A^{(1)} = iA_1$$

where A_1 is Hermitian. Hence

$$\lambda(A^{(1)}) = i\mu(A^{(1)}), \tag{39}$$

where $\mu(A^{(1)}) = \lambda(A_1)$ is a real number (as $\lambda(A)$ is real on $\mathcal{H}_{\nu+1}$) which is $\not\equiv 0$ on $\mathcal{H}_{\nu+1}$ (by our assumption that $\lambda(A^{(1)}) \not\equiv 0$). We show that also in this case relations (37) hold; then repeating the argument used in case α) we also get a contradiction, proving that $\lambda(A^{(1)}) \not\equiv 0$ is impossible also in case β).

By (36), for $p=1$ we have

$$Ax_1 = \lambda(A)x_1 + \lambda(A^{(1)})x_0,$$

hence

$$(Ax_1, x_0) = \lambda(A)(x_1, x_0) + \lambda(A^{(1)})(x_0, x_0).$$

On the other hand, if $A \in \mathcal{H}_{\nu+1}$, we have

$$(Ax_1, x_0) = (x_1, Ax_0) = (x_1, \lambda(A)x_0) = \lambda(A)(x_1, x_0),$$

hence

$$\lambda(A^{(1)})(x_0, x_0) = 0.$$

As $\lambda(A^{(1)}) \not\equiv 0$ we have $(x_0, x_0) = 0$ and so (37) holds for $q+r = 0$. Now we suppose that (37) holds for $q+r<p$ and prove it to be true for $q+r = p$. To this end we take the inner product of both sides of (36) with x_1. In virtue of our inductive assumption we get

$$(Ax_p, x_1) = \lambda(A)(x_p, x_1) + p\lambda(A^{(1)})(x_{p-1}, x_1). \tag{40}$$

On the other hand if $A \in \mathcal{H}_{\nu+1}$ we have in virtue of (38) and (39)

$$\begin{aligned}(Ax_p, x_1) &= (x_p, Ax_1) = (x_p, \lambda(A)x_1 + \lambda(A^{(1)})x_0) = \\ &= \lambda(A)(x_p, x_1) + \overline{\lambda(A^{(1)})}(x_p, x_0) = \lambda(A)(x_p, x_1) - \lambda(A^{(1)})(x_{p-1}, x_1)\end{aligned} \tag{41}$$

and comparing (40) and (41) we see that

$$(p+1)\lambda(A^{(1)})(x_{p-1}, x_1) = 0.$$

As $p+1>0$, $\lambda(A^{(1)}) \not\equiv 0$ we must have $(x_{p-1}, x_2) = 0$ concluding the proof of (37).

So we have proved that in every case $\lambda(A^{(1)}) \equiv 0$ and relations (36) take the form

$$Ax_p = \lambda(A)x_p \quad \text{for} \quad p = 0, 1, 2, \ldots \quad \text{and} \quad A \in X_{p+1}.$$

Hence we have also on the closed subspace $\mathfrak{M}$ generated by x_p $(p=0, 1, 2, \ldots)$

$$(42) \qquad Ax=\lambda(A)x \text{ for all } x\in\mathfrak{M} \text{ and } A\in X_{\nu+1};$$

on the other hand $\mathfrak{M}$ is invariant with respect to the Hermitian operator H_ν. The subspace $\mathfrak{M}$ contains the non-negative vector $x_0\neq 0$; hence only the following three cases $\alpha')-\gamma')$ are possible:

α') $\mathfrak{M}$ is non-negative. Then $\dim\mathfrak{M}\leqq k$ and H_ν has an eigenvector $y\neq 0$ in $\mathfrak{M}$, which in virtue of (42) is also an eigenvector of all $A\in X_{\nu+1}$; by condition b) of the theorem, y is a common eigenvector for all $A\in X_\nu$ and y is non-negative as $\mathfrak{M}$ is non-negative.

β') (x, x) changes its sign on $\mathfrak{M}$ and the inner product (x, y) is non-degenerate on $\mathfrak{M}$. Then $\mathfrak{M}$ is a space $\Pi_{\varkappa'}$ and by PONTRYAGIN's theorem (see also Corollary 1) $\mathfrak{M}$ has a $\varkappa'$-dimensional non-negative subspace $\mathfrak{N}$ which is invariant with respect to H_ν. Let $y\neq 0$ be an eigenvector of H_ν in $\mathfrak{N}$; then y is non-negative and by (42) it is also an eigenvector for all $A\in X_{\nu+1}$. Hence by condition b) it is also a common eigenvector for all $A\in X_\nu$.

γ') (x, x) changes its sign on $\mathfrak{M}$ and the scalar product (x, y) degenerates on $\mathfrak{M}$. Let $\mathfrak{N}$ be the isotropic subspace of $\mathfrak{M}$, i. e. $\mathfrak{N} = \mathfrak{M}\cap\mathfrak{M}^\perp$. As $\mathfrak{M}$ is invariant with respect to H_ν, the subspaces $\mathfrak{M}^\perp$ and $\mathfrak{N}$ have the same property.

But $\mathfrak{N}$ is a nullspace, hence $\dim\mathfrak{N}\leqq\varkappa$ and therefore H_ν has an eigenvector $y\neq 0$ in $\mathfrak{N}$. Repeating the argument at the end of β') we see, that y is a common non-negative eigenvector of all $A\in X_{\nu+1}$. This concludes the proof of assertion 2).

Assertion 2) means that a non-negative subspace (of dimension $\geqq 1$ and $\leqq\varkappa$) exists which is invariant with respect to all $A\in X$. Using this fact and repeating the argument in the proof of proposition VI we see that if $\dim\mathfrak{N}<\varkappa$, then $\mathfrak{N}\subset\mathfrak{N}_1$, $\mathfrak{N}\neq\mathfrak{N}_1$, where $\mathfrak{N}_1$ is also non-negative and invariant with respect to all $A\in X$. As in the proof of theorem 1, this proves assertion 1) of theorem 2.

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Über Semiringe mit multiplikativer Kürzungsregel

Von O. STEINFELD in Budapest

Herrn Professor Béla Szőkefalvi-Nagy zum 50. Geburtstag gewidmet

§ 1

Unter einem *Semiring* verstehen wir eine (nichtleere) Menge $S=\{\alpha, \beta, \gamma, \ldots\}$, in der eine Addition und eine Multiplikation mit den folgenden Eigenschaften definiert sind: (i) S ist eine additive Halbgruppe, (ii) S ist eine multiplikative Halbgruppe, (iii) die links- und rechtsseitigen Distributivitätsregeln $\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma$ und $(\alpha+\beta)\gamma = \alpha\gamma+\beta\gamma$ sind auch gültig.

Das Element 0 von S heißt *Nullelement*, wenn

$$0+\xi = \xi+0 = \xi \quad \text{und} \quad 0\xi = \xi 0 = 0$$

für jedes $\xi (\in S)$ gelten.

Ein Semiring enthält höchstens ein Nullelement. Offenbar kann man zu jedem Semiring ein Nullelement adjungieren.

Ist die Addition in einem Semiring S kommutativ, und besitzt S ein Nullelement, so nennen wir S einen *Halbring*.

Wir sagen, daß in einem Semiring S die *linksseitige multiplikative Kürzungsregel* gilt, wenn

$$\sigma\alpha=\sigma\beta \Rightarrow \alpha=\beta \qquad (\alpha, \beta, \sigma \in S;\ \sigma \neq 0) \tag{1}$$

für jedes $\sigma (\neq 0, \sigma \in S)$ gilt.

Ähnlich definiert man die *rechtsseitige multiplikative Kürzungsregel* in einem Semiring.

Wir sagen, daß in einem *Semiring die multiplikative Kürzungsregel* gilt, wenn in ihm die linksseitige und rechtsseitige multiplikative Kürzungsregel gleichzeitig gelten.

Ein Semiring mit multiplikativer Kürzungsregel ist offenbar nullteilerfrei. Die Umkehrung dieser Behauptung ist dagegen ungültig, siehe z. B. BOURNE [1], Beispiele 2 und 3.

Es ist bekannt, daß die Charakteristik eines nullteilerfreien Ringes mit mindestens zwei Elementen 0 oder eine Primzahl ist. (Siehe z. B. RÉDEI [2], Satz 38.)

Wir wollen in dieser Arbeit ein Analogon dieses Resultates für Semiringe mit linksseitiger (rechtsseitiger) multiplikativer Kürzungsregel beweisen. (S. Satz 1.)

J. SZENDREI hat in seiner Arbeit [4] bewiesen, daß es zu jedem nullteilerfreien Ring einen ebensolchen „minimalen" Oberring mit Einselement gibt.

In Satz 2 werden wir beweisen, daß ein ähnliches Ergebnis über Halbringe mit multiplikativer Kürzungsregel gilt.

§ 2

Zur Vorbereitung betrachten wir nun eine multiplikative Halbgruppe $H=\{\alpha, \beta, \gamma, \ldots\}$.

Wir sagen, daß ein Element $\alpha(\in H)$ von *endlicher Ordnung* bzw. *unendlicher Ordnung* ist, je nachdem unter den Potenzen $\alpha, \alpha^2, \ldots$ nur endlich viele oder unendlich viele verschiedene Elemente vorkommen. Im ersten Fall bezeichnen wir mit $o(\alpha)$ die Anzahl der verschiedenen Elemente unter den Potenzen $\alpha, \alpha^2, \ldots$. Es ist bekannt (s. z. B. RÉDEI [2], § 20):

Ist α von unendlicher Ordnung, so besteht die Folge $\alpha, \alpha^2, \ldots$ aus lauter verschiedenen Gliedern. Im Falle $o(\alpha)=n$ ist die Folge $\alpha, \alpha^2, \ldots$ stets von der Form

$$\alpha, \alpha^2, \ldots, \alpha^{k-1}, \alpha^k, \ldots, \alpha^n; \ \alpha^{n+1}=\alpha^k, \ldots, \alpha^{2n}=\alpha^{n+k-1} \qquad (1 \leqq k \leqq n), \tag{2}$$

wobei $\alpha, \ldots, \alpha^n$ verschieden sind. Die Elemente $\alpha^k, \alpha^{k+1}, \ldots, \alpha^n$ bilden eine zyklische Gruppe $\langle\alpha\rangle$ von der Ordnung $n-k+1$.

Es folgt aus den obigen, daß man zu jedem Element $\alpha(\in H)$ von endlicher Ordnung zwei eindeutig bestimmte natürliche Zahlen $o(\alpha)=n$ und $s(\alpha)=k$ zuordnen kann, wobei $o(\alpha)=n$ die Ordnung von α und $s(\alpha)=k$ die *Sprungstelle* von α d. h. die kleinste natürliche Zahl k mit $\alpha^{n+1}=\alpha^k$ $(1 \leqq k \leqq n)$ bezeichnen.

Das folgende Ergebnis ist unseres Wissens neu und es stammt im wesentlichen von A. RÉNYI.

Hilfssatz. *Hat das Element α einer Halbgruppe H* die *Ordnung $o(\alpha)=n$*H *und die Sprungstelle $s(\alpha)=k$, so gilt*

$$o(\alpha^r) = m+\frac{n-k+1}{(n-k+1, r)} \quad \text{und} \quad s(\alpha^r) = m+1 \qquad (1 \leqq r \leqq n), \tag{3}$$

wobei $k = mr+j$ $(0 \leqq m;\ 1 \leqq j \leqq r)$ besteht.

Beweis. Nach der Voraussetzung ist $m+1$ die kleinste natürliche Zahl mit $(m+1)r \geqq k$, deshalb ist $\alpha^{(m+1)r}=\alpha^{k+u}$ $(0 \leqq u \leqq n-k)$ ein Element der Gruppe $\langle\alpha\rangle$. Wir haben die kleinste natürliche Zahl x mit $\alpha^{k+u+xr}=\alpha^{k+u}$ zu bestimmen. Da $\alpha^{k+u+xr}=\alpha^{k+u} \Leftrightarrow k+u+xr \equiv k+u \pmod{n-k+1}$ gilt, besteht $x=\dfrac{n-k+1}{(n-k+1, r)}$.

Damit ist der Hilfssatz bewiesen.

§ 3

In einem Semiring kann man über die *additive Ordnung* $o^+(\alpha)$ und über die *multiplikative Ordnung* $o^\times(\alpha)$ ferner über die *additive Sprungstelle* $s^+(\alpha)$ und über die *multiplikative Sprungstelle* $s^\times(\alpha)$ eines Elementes sprechen.

Gilt $o^+(\alpha)=\infty$ in einem Semiring S für alle $\alpha(\neq 0)$, so verstehen wir unter der *Charakteristik* von S die Zahl 0. Gibt es dagegen eine natürliche Zahl n, so daß stets $o^+(\alpha)=m$ und $m|n$ gelten, dann nennen wir die kleinste solche Zahl n die Charakteristik von S. In jedem anderen Fall habe S die Charakteristik ∞.

Satz 1. *Die Charakteristik eines Semiringes mit linksseitiger (rechtsseitiger) multiplikativer Kürzungsregel ist entweder* 0 *oder* 1, *oder aber eine Primzahl* p. *Im letzteren Falle hat jedes Element* $\alpha(\neq 0)$ *die additive Sprungstelle* 1, *d. h. die Elemente* $\alpha, 2\alpha, \ldots, p\alpha$ *bilden eine* p-*Gruppe*.

Beweis. Ist die additive Ordnung jedes Elementes $\alpha(\neq 0)$ von S unendlich, so ist nichts zu beweisen.

Es sei $\alpha(\neq 0)$ ein Element von der endlichen, additiven Ordnung $o^+(\alpha)=n$ und von der additiven Sprungstelle $s^+(\alpha)=k$. Wir zeigen, daß jedes Element $\beta(\neq 0)$ von S dieselbe additive Ordnung n und dieselbe additive Sprungstelle k hat.

Betrachten wir die Elemente $\beta, 2\beta, \ldots, n\beta$. Wäre

$$m\beta=l\beta \qquad (1\leqq l<m\leqq n)$$

gültig, so bestände

$$m\beta\alpha=l\beta\alpha=\beta m\alpha=\beta l\alpha \Rightarrow m\alpha=l\alpha \qquad (1\leqq l<m\leqq n),$$

was der Annahme $o^+(\alpha)=n$ widerspricht. So gilt $o^+(\alpha)\leqq o^+(\beta)$. Wir werden einsehen, daß der Fall $o^+(\alpha)<o^+(\beta)$ unmöglich ist. Wären nämlich die Elemente $\beta, 2\beta, \ldots, n\beta, \ldots, r\beta (r>n)$ alle verschieden, so wäre wegen $o^+(\alpha)=n$

$$r\alpha=s\alpha \qquad (1\leqq s\leqq n<r)$$

gültig, woraus wegen $r\alpha\beta=s\alpha\beta=\alpha r\beta=\alpha s\beta$ der Widerspruch $r\beta=s\beta$ $(1\leqq s<r)$ folgte.

Damit ist

$$o^+(\alpha)=o^+(\beta)=n \qquad (\text{für jedes } \beta\neq 0, \in S) \tag{4}$$

bewiesen.

Da man die Äquivalenz $(n+1)\alpha=k\alpha \Leftrightarrow (n+1)\beta=k\beta$ leicht einsehen kann, ist auch $s^+(\alpha)=s^+(\beta)$ richtig.

Es wird gezeigt, daß $o^+(\alpha)=n$ entweder 1 oder eine Primzahl p sein muß. Im entgegengesetzten Falle wäre $p|n$ ($p<n$, p eine Primzahl) gültig. Man bekommt aus dem Hilfssatz

$$o^+(p\alpha)\leqq n-1 \qquad \text{für jede Sprungstelle } s^+(\alpha). \tag{5}$$

Da wegen $o^+(\alpha)=n>p$ das Element $p\alpha$ von Null verschieden ist, steht (5) im Widerspruch mit (4).

Wir haben noch zu beweisen, daß im Falle $o^+(\alpha)=p$ die Behauptung $s^+(\alpha)=1$ besteht. Wäre nämlich $s^+(\alpha)=k\geqq 2$, so folgte aus dem Hilfssatz $s^+(k\alpha)=1\neq k$, was nach den Vorigen unmöglich ist.

Damit ist der Beweis von Satz 1 beendet.

§ 4

Wir schicken einige Begriffe voraus.

Eine additive Teilhalbgruppe $\mathfrak{a}$ eines Semiringes S wird ein *Ideal* von S genannt, wenn für alle Elemente $\alpha \in \mathfrak{a}$ und $\sigma \in S$

$$\alpha\sigma, \sigma\alpha \in \mathfrak{a}$$

gilt.

Wir sagen, daß eine Klasseneinteilung $\mathfrak{C}$ eines Semiringes S *kompatibel* ist, wenn die der Klasseneinteilung $\mathfrak{C}$ zugehörige Äquivalenzrelation $\equiv$ eine *Kongruenzrelation* ist, d. h. wenn für die Elemente $\varkappa, \lambda, \sigma \in S$ die Regeln

$$\varkappa \equiv \lambda \pmod{\mathfrak{C}} \Rightarrow \varkappa + \sigma \equiv \lambda + \sigma \pmod{\mathfrak{C}}, \quad \sigma + \varkappa \equiv \sigma + \lambda \pmod{\mathfrak{C}} \tag{6}$$

und

$$\varkappa \equiv \lambda \pmod{\mathfrak{C}} \Rightarrow \varkappa\sigma \equiv \lambda\sigma \pmod{\mathfrak{C}}, \quad \sigma\varkappa \equiv \sigma\lambda \pmod{\mathfrak{C}} \tag{7}$$

gelten. (S. z. B. RÉDEI [2], §§ 30 und 46.)

Wir betrachten eine kompatible Klasseneinteilung $\mathfrak{C}$ eines Semiringes S. Die durch das Element $\alpha (\in S)$ repräsentierte Klasse und die Menge der Klassen bezeichnen wir mit $\bar{\alpha}$ bzw. $\bar{S}$. Definiert man in $\bar{S}$ die Verknüpfungen

$$\bar{\alpha} + \bar{\beta} = \overline{\alpha + \beta} \quad \text{und} \quad \bar{\alpha}\bar{\beta} = \overline{\alpha\beta} \qquad (\alpha, \beta \in S), \tag{8}$$

so wird $\bar{S}$ ein Semiring, den wir einen *Faktorsemiring* nennen.

Wir bezeichnen mit N den Halbring der nichtnegativen ganzen Zahlen.

Satz 2 (Vgl. SZENDREI [4] und RÉDEI [2], Satz 201). *Jeder Halbring*[1]) $R = \{\alpha, \beta, \gamma, \ldots\}$ *mit multiplikativer Kürzungsregel hat ebensolche Erweiterungshalbringe mit Einselement. Unter ihnen gibt es einen, der mit S bezeichnet werde, derart, daß die übrigen einen mit S isomorphen Teilhalbring enthalten. Dabei ist R ein Ideal von S.*

Beweis. Es bezeichne ω das Nullelement von R. Wir betrachten die Menge R_1 aller Elementenpaare (a, α), wo a bzw. α die Elemente von N bzw. von R durchlaufen. Definiert man in R_1 die Verknüpfungen

$$(a, \alpha) + (b, \beta) = (a + b, \alpha + \beta) \qquad (a, b \in \mathsf{N};\ \alpha, \beta \in R), \tag{9}$$

$$(a, \alpha)(b, \beta) = (ab, a\beta + b\alpha + \alpha\beta) \qquad (a, b \in \mathsf{N};\ \alpha, \beta \in R), \tag{10}$$

so ist es leicht einzusehen, daß R_1 ein Halbring mit dem Einselement $(1, \omega)$ ist und die Elemente der Form $(0, \xi)$ ein Ideal von R_1 bilden, welches mit dem Halbring R isomorph ist[2]).

Wir schicken die folgende *Bemerkung* voraus:

Gilt für die Elemente $(m, \mu), (n, \nu)$ und $(0, \alpha)$ $(\alpha \neq \omega)$ eine der Bedingungen

$$(m, \mu)(0, \alpha) = (n, \nu)(0, \alpha); \quad (0, \alpha)(m, \mu) = (0, \alpha)(n, \nu), \tag{11}$$

[1]) Die Existenz des Nullelementes ist nur der Einfachkeit halber vorausgesetzt.

[2]) Diese Erweiterung ist das Analogon der Dorrohschen Ringerweiterung.

so gelten

(12) $$(m, \mu)(0, \xi) = (n, \nu)(0, \xi) \text{ und } (0, \xi)(m, \mu) = (0, \xi)(n, \nu)$$

für jedes Element $(0, \xi)$ $(\xi \in R)$.

Aus (10) und (11) folgt nämlich $(0, m\xi + \xi\mu)(0, \alpha) = \big((0, \xi)(m, \mu)\big)(0, \alpha) = \big((0, \xi)(n, \nu)\big)(0, \alpha) = (0, n\xi + \xi\nu)(0, \alpha)$. Daraus bekommt man nach Kürzung mit dem Element $(0, \alpha)$ die Behauptung (12_2). Ähnlicherweise ist (12_1) einzusehen.

Wir betrachten die folgende Klasseneinteilung C des Halbringes R_1

(13) $$\begin{cases} (r, \varrho) \equiv (s, \sigma) \,(\mathrm{mod}\ \mathsf{C}) \Leftrightarrow (r, \varrho)(0, \alpha) = (s, \sigma)(0, \alpha) \\ \text{für ein gegebenes Element } (0, \alpha) \text{ mit } \alpha \neq \omega. \end{cases}$$

Um zu zeigen, daß C eine kompatible Klasseneinteilung von R_1 ist, haben wir die Erfüllung der Regeln (6), (7) einzusehen.

Ist $(r, \varrho) \equiv (s, \sigma)$ (mod C), so gilt für ein beliebiges $(b, \beta) \in R_1$ die Regel $\big((r, \varrho) + (b, \beta)\big)(0, \alpha) = (r, \varrho)(0, \alpha) + (b, \beta)(0, \alpha) = (s, \sigma)(0, \alpha) + (b, \beta)(0, \alpha) = \big((s, \sigma) + (b, \beta)\big)(0, \alpha)$ d. h.

$$(r, \varrho) + (b, \beta) \equiv (s, \sigma) + (b, \beta) \,(\mathrm{mod}\ \mathsf{C}).$$

Da ferner auch

$$(b, \beta)(r, \varrho)(0, \alpha) = (b, \beta)(s, \sigma)(0, \alpha) \Rightarrow (b, \beta)(r, \varrho) \equiv (b, \beta)(s, \sigma) \,(\mathrm{mod}\ \mathsf{C})$$

und nach der vorausgeschickten Bemerkung

$$\big((r, \varrho)(b, \beta)\big)(0, \alpha) = (r, \varrho)\big((b, \beta)(0, \alpha)\big) = (s, \sigma)\big((b, \beta)(0, \alpha)\big) =$$
$$= \big((s, \sigma)(b, \beta)\big)(0, \alpha) \Rightarrow (r, \varrho)(b, \beta) \equiv (s, \sigma)(b, \beta) \,(\mathrm{mod}\ \mathsf{C})$$

gelten, sind (6) und (7) für C bewiesen. Es bezeichne $\overline{(r, \varrho)}$ bzw. $\overline{R_1} = S$ die durch das Element (r, ϱ) $(\in R_1)$ repräsentierte Klasse bzw. den Faktorhalbring.

Offenbar ist $\overline{(1, \omega)}$ das Einselement von S.

Da die Elemente der Form $(0, \xi)$ $(\xi \in R)$ einen mit R isomorphen Halbring, also einen Halbring mit multiplikativer Kürzungsregel bilden, gilt nach (13)

(14) $$(0, \xi) \equiv (0, \tau) \,(\mathrm{mod}\ \mathsf{C}) \Leftrightarrow (0, \xi) = (0, \tau).$$

So bilden die Klassen der Form $\overline{(0, \xi)}$ einen mit R isomorphen Teilhalbring, sogar ein Ideal S^* von S.

Aus (13) folgt

(15) $$\overline{(c, \gamma)} = \overline{(0, \omega)} \Leftrightarrow (c, \gamma)(0, \alpha) = (0, \omega).$$

Wir haben noch zu zeigen, daß die multiplikative Kürzungsregel in S gilt. Es sei

(16) $$\overline{(r, \varrho)}\,\overline{(b, \beta)} = \overline{(s, \sigma)}\,\overline{(b, \beta)} \qquad \big(\overline{(b, \beta)} \neq \overline{(0, \omega)}\big).$$

Nach (8_2), (13) und (15) folgt aus (16)

(17) $$(r, \varrho)(b, \beta)(0, \alpha) = (s, \sigma)(b, \beta)(0, \alpha) \qquad \big((b, \beta)(0, \alpha) \neq (0, \omega)\big).$$

Dies impliziert $(0, \alpha)(r, \varrho)\cdot(b, \beta)(0, \alpha) = (0, \alpha)(s, \sigma)\cdot(b, \beta)(0, \alpha)$ $\big((0, \alpha)(r, \varrho), (0, \alpha)(s, \sigma), (b, \beta)(0, \alpha) \in S^*\big)$. Da in S^* die multiplikative Kürzungsregel gilt, besteht $(0, \alpha)(r, \varrho) = (0, \alpha)(s, \sigma)$. Nach der vorausgeschickten Bemerkung und (13) bekommt man daraus

$$\overline{(r, \varrho)} = \overline{(s, \sigma)}. \tag{18}$$

Durch ähnliche Methode wird die Implikation

$$\overline{(b, \beta)}\,\overline{(r, \varrho)} = \overline{(b, \beta)}\,\overline{(s, \sigma)} \Rightarrow \overline{(r, \varrho)} = \overline{(s, \sigma)} \qquad (\overline{(b, \beta)} \neq \overline{(0, \omega)})$$

bewiesen.

Es sei endlich T ein Erweiterungshalbring von R mit dem Einselement ε, in dem auch die multiplikative Kürzungsregel erfüllt ist. Man kann voraussetzen, daß selbst R ein Teilhalbring von T ist. Wir zeigen, daß

$$\overline{(s, \sigma)} \to s\varepsilon + \sigma \qquad (s \in N,\ \sigma \in R) \tag{19}$$

eine isomorphe Abbildung von S in T liefert.

Wegen

$$\overline{(s, \sigma)} + \overline{(t, \tau)} = \overline{(s+t,\ \sigma+\tau)} \to (s+t)\varepsilon + (\sigma+\tau) = s\varepsilon + \sigma + t\varepsilon + \tau$$

und

$$\overline{(s, \sigma)}\,\overline{(t, \tau)} = \overline{(st,\ s\tau + t\sigma + \sigma\tau)} \to st\varepsilon + s\tau + t\sigma + \sigma\tau = (s\varepsilon + \sigma)(t\varepsilon + \tau)$$

ist (19) eine homomorphe Abbildung.

Um die Eineindeutigkeit einzusehen, setzen wir voraus, daß $s\varepsilon + \sigma = t\varepsilon + \tau$ gilt. Multipliziert man diese Gleichung von rechts mit dem Element $\alpha \neq \omega$ von R, so entsteht $s\alpha + \sigma\alpha = t\alpha + \tau\alpha$. Dies bedeutet die Gültigkeit der Bedingung $(s, \sigma)(0, \alpha) = (t, \tau)(0, \alpha)$, woraus nach (13)

$$\overline{(s, \sigma)} = \overline{(t, \tau)}$$

folgt[3].

Damit ist der Beweis beendet.

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[3]) Vgl. den Beweis des Satzes von Szendrei in Rédei [3].

Permutations in finite fields

By L. CARLITZ in Durham (N. C., U. S. A)

1. A polynomial $f(x)$ with coefficients in the finite field $GF(q)$ is called a permutation polynomial if the numbers $f(a)$, where $a \in GF(q)$, are a permutation of the a's. That such polynomials exist is evident from the Lagrange interpolation formula for a finite field:

$$f(x) = -\sum_a \frac{x^q - x}{x-a} f(a). \tag{1.1}$$

The formula (1. 1) furnishes a polynomial that is of degree $<q$. We shall say generally that a permutation polynomial is in *reduced* form when its degree $<q$. It is known that for $q>2$ permutation polynomials of degree $q-1$ cannot occur; more precisely the degree of a non-linear permutation polynomial cannot be a divisor of $q-1$. This follows very easily from

$$\sum_{a \in GF(q)} a^k = \begin{cases} 0 & (0 \leqq k < q-1) \\ -1 & (k = q-1), \end{cases} \tag{1.2}$$

Assume that

$$f(x) = c_0 x^m + \dots + c_m \quad (c_j \in GF(q),\ c_0 \neq 0)$$

is a permutation polynomial and that $q-1 = mr$. Then

$$(f(x))^r = c_0^r x^{mr} + \dots + c_m^r$$

so that

$$0 = \sum_{a \in GF(q)} (f(a))^r = -c_0^r.$$

This contradicts $c_0 \neq 0$.

Dickson [3] has constructed various classes of permutation polynomials. Rédei [5] has considered rational functions over $GF(q)$ that possess an inverse. He has proved in particular that if m is odd, $1 \leqq m < q$, then there exist rational permutation functions of degree m.

The writer [2] has proved that every permutation polynomial is generated by the special polynomials

$$ax+b,\ x^{q-2} \quad (a, b \in GF(q),\ a \neq 0). \tag{1.3}$$

For $q=5$ this had been proved by Betti and for $q=7$ by Dickson [3, p. 119].

Clearly if $f(x)$ is a permutation polynomial for $GF(q)$, the same is true for $f(x)+(x^q-x)g(x)$, where $g(x)$ is an arbitrary polynomial with coefficients in $GF(q)$. Indeed the theorem quoted above is to be understood in this sense. Thus if $f(x)$

is a permutation polynomial in reduced form then

$$F(x) = f(x) + (x^q - x)g(x), \tag{1. 4}$$

where $F(x)$ is the resultant of a finite number of the special permutations (1. 3) and $g(x)$ is some polynomial in $GF[q, x]$. We may call $F(x)$ a *crude* permutation polynomial. Note in particular that in computing the polynomial $F(x)$ reduction (mod $x^q - x$) is not allowed. Also $F(x)$ is not uniquely determined by $f(x)$. For example the polynomials

$$x^{(q-2)2r} \qquad (r = 1, 2, 3, \ldots)$$

are all crude permutation polynomials corresponding to the polynomial x.

2. Now let $f(x)$ be a permutation polynomial for $GF(q)$ in reduced form. It is of interest to ask whether there exist polynomials congruent to $f(x)$ (mod $x^q - x$) that are also permutation polynomials for $GF(q^r)$ where r is assigned. We first prove the following result.

Theorem 1. *Let $f(x)$ be a permutation polynomial for $GF(q)$ in reduced form of degree >1 and let $F(x)$ be a crude permutation polynomial corresponding to $f(x)$. Then $F(x)$ is a permutation polynomial for $GF(q^r)$ if and only if*

$$(2^r - 1, q - 2) = 1. \tag{2. 1}$$

Since $\deg f(x) > 1$ we have also $\deg F(x) > 1$. Consequently the permutation x^{q-2} occurs at least once in $F(x)$. Now x^{q-2} effexts a permutation in $GF(q^r)$ if and only if

$$(q^r - 1, q - 2) = 1. \tag{2. 2}$$

Since $q^r - 1 \equiv 2^r - 1 \pmod{q-2}$, it follows that the condition (2. 2) is equivalent to (2. 1). This evidently completes the proof of the theorem.

Suppose that q is odd and greater than 3. Let 2 belong to the exponent t (mod $q-2$). Then (2. 1) is certainly satisfied when $r \equiv 1 \pmod t$ but is not satisfied when $r \equiv 0 \pmod t$. When q is even and greater than 4, let 2 belong to the exponent t $\left(\text{mod } \frac{1}{2}(q-2)\right)$. Then again (2. 1) is satisfied when $r \equiv 1 \pmod t$ and not satisfied when $r \equiv 0 \pmod t$. We have therefore

Theorem 2. *Let $F(x)$ be a crude permutation polynomial for $GF(q)$. Then if $q > 4$ there are infinitely many $GF(q^r)$ for which $F(x)$ is a permutation polynomial and also infinitely many $GF(q^r)$ for which $F(x)$ is not a permutation polynomial.*

When $q = 4$, x^2 is a permutation polynomial for all $GF(2^r)$. When $q = 3$ the special permutations (1. 3) are all of the first degree.

3. Put $q = p^n$, where p is a prime. Then it is easily verified that the polynomial

$$ax^{p^j} + b \qquad (a, b \in GF(q),\ a \neq 0) \tag{3. 1}$$

is a permutation polynomial for all $GF(q^r)$ and for all $j = 0, 1, 2, \ldots$.

If $f(x)$ is an arbitrary permutation polynomial for $GF(q)$ then for every $c \in GF(q)$ the equation $f(x) = c$ is solvable in $GF(q)$ and indeed has a unique solution $b \in GF(q)$.

Assume $f(x) \in GF[q, x]$; then

$$f(x) - c = (x-b)^k M(x), \tag{3.2}$$

where $k \geqq 1$, $M(x) \in GF[q, x]$ and either $\deg M(x) = 0$ or $M(x)$ is a product of irreducible polynomials $P_i(x) \in GF[q, x]$, $\deg P_i(x) \geqq 2$. Hence if r is a multiple of any $d_i = \deg P_i(x)$ it follows at once from (3. 2) that $f(x)$ is not a permutation polynomial for $GF(q^r)$. We accordingly suppose that (3. 2) reduces to

$$f(x) - c = a(x-b)^k \qquad (a \neq 0); \tag{3.3}$$

that is for each $c \in GF(q)$ there is $ab = b(c) \in GF(q)$ such that (3. 3) holds. In particular for $c = 1, 0$, (3. 3) implies

$$a(x-b_0)^k - a(x-b_1)^k = 1. \tag{3.4}$$

Replacing x by $x + b_1$, (3. 4) becomes

$$a(x+b)^k - ax^k = 1 \qquad (b = b_1 - b_0).$$

Expanding by the binomial theorem we get

$$\binom{k}{s} \equiv 0 \pmod{p} \qquad (0 < s < k). \tag{3.5}$$

By a known property of binomial coefficients it follows that $k = p^j$ for some j. We have therefore proved the following

Theorem 3. *A polynomial $f(x) \in GF[q, x]$ is a permutation polynomial for all $GF(q^r)$ if and only if it is of the form* (3. 1).

We have incidently proved the following result.

Theorem 4. *If $f(x)$ is a permutation polynomial for $GF(q)$ that is not of the form* (3. 1), *then for infinitely many r, $f(x)$ is not a permutation polynomial for $GF(q^r)$.*

It might seem plausible that if $f(x)$ is a permutation polynomial for $GF(q)$ then it will also be a permutation for infinitely many $GF(q^r)$. We have seen that this is true for crude permutation polynomials (Theorem 2). Two other classes of polynomials with this property are covered by the following two theorems.

Theorem 5. *Let $(k, q-1) = 1$ so that x^k is a permutation polynomial for $GF(q)$. Then there are infinitely many $GF(q^r)$ for which x^k is a permutation polynomial and infinitely many $GF(q^r)$ for which x^k is not a permutation polynomial.*

There is no loss in generality in assuming that $(k, q) = 1$. Let q belong to the exponent $t \pmod{k}$, so that $t > 1$. Then for r divisible by t we have $q^r \equiv 1 \pmod{k}$, so that x^k is certainly not a permutation polynomial for $GF(q^r)$. On the other hand for $r \equiv 1 \pmod{t}$ we have

$$q^r - 1 \equiv q - 1 \pmod{k},$$

so that $(k, q^r - 1) = (k, q-1) = 1$. Hence x^k is a permutation polynomial for all $GF(q^{mt+1})$, $m = 1, 2, 3, \ldots$.

Theorem 6. *Let $q=p^n$ and put*

(3. 6) $$f(x) = c_0x + c_1x^p + \dots + c_{n-1}x^{p^{n-1}} \qquad (c_j \in GF(p)).$$

Then $f(x)$ is a permutation polynomial for $GF(q)$ if and only if

(3. 7) $$(c_0 + c_1x + \dots + c_{n-1}x^{n-1}, 1-x^n) = 1.$$

Moreover there are infinitely many $GF(q^r)$ for which $f(x)$ is a permutation polynomial and infinitely many $GF(q^r)$ for which $f(x)$ is not a permutation polynomial.

The first part of the theorem is a corollary of the existence of a normal basis for $GF(q)$; see for example [4, p. 250].

To prove the second part put

$$C(x) = c_0 + c_1x + \dots + c_{n-1}x^{n-1}.$$

Then $f(x)$ is a permutation polynomial for $GF(q^r)$ if and only if

(3. 8) $$(C(x), 1-x^{rn}) = 1.$$

There is no loss in generality in assuming that $c_0 \neq 0$, so that $(x, C(x))=1$. Now let x belong to the exponent $t \pmod{C(x)}$. Then for $r \equiv 1 \pmod t$ we have

$$1-x^{rn} \equiv 1-x^n \pmod{C(x)},$$

so that $(C(x), 1-x^{rn}) = (C(x), 1-x^n) = 1$; clearly $f(x)$ is a permutation polynomial for $GF(q^r)$. On the other hand if $r \equiv 0 \pmod t$, then $1-x^{rn} \equiv 0 \pmod{C(x)}$ and it follows that $f(x)$ is not a permutation polynomial for such $GF(q^r)$. This completes the proof of the theorem.

4. Dickson [3] showed that the quartic

(4. 1) $$f(x) = x^4 + 3x$$

is a permutation polynomial for $GF(7)$ but not for any $GF(7^n)$, $n>1$. This result can be generalized as follows.

Put $q = 2m+1$. We shall show that for proper choice of $a \in GF(q)$ the polynomial

(4. 2) $$f(x) = x^{m+1} + ax$$

is a permutation polynomial for $GF(q)$.

It is convenient to define

(4. 3) $$\psi(x) = x^m.$$

Thus $\psi(c) = 1, -1$ or 0 according as c is a non-zero square, a non-square or zero in $GF(q)$. We may rewrite (4. 2) as

(4. 4) $$f(x) = x(a+\psi(x)).$$

We assume that $a^2 \neq 1$ so that $f(x)$ vanishes only when $x=0$. Now if $f(x)$ is not a permutation polynomial we must have

(4. 5) $$f(b) = f(c) \qquad (b, c \in GF(q),\ b \neq c,\ bc \neq 0)$$

for at least one pair b, c. We consider two cases (i) $\psi(b) = \psi(c)$, (ii) $\psi(b) = -\psi(c)$.

In case (i) it follows from (4. 4) and (4. 5) that

$$b(a+\psi(b)) = c(a+\psi(b));$$

since $a^2 \neq 1$, it follows that $b=c$.

In case (ii) we get similarly

$$b(a+\psi(b)) = c(a-\psi(b)),$$

so that
$$-1 = \psi(bc) = \psi\left(\frac{a+1}{a-1}\right).$$

Hence if we choose a so that

$$\psi\left(\frac{a+1}{a-1}\right) = 1 \tag{4. 6}$$

we have a contradiction. Clearly (4. 6) can be satisfied by taking

$$a = (u^2+1)/(u^2-1), \tag{4. 7}$$

where u^2 is an arbitrary square of the field (different from $\pm 1, 0$). For $q \geqq 7$ such u^u always exist. The value of a furnished by (4. 7) automatically satisfies the condition $a^2 \neq \pm 1$.

This proves the following

Theorem 7. *For $q = 2m+1 \geqq 7$, the polynomial* (4. 2) *is a permutation polynomial for $GF(q)$ provided that a is defined by* (4. 7) *with u^2 an arbitrary square of $GF(q)$ different from* 1, 0.

For $q=7, u^2=2$, it is easily verified that (4. 2) reduces to (4. 1).

It can be proved that if k is a fixed integer $\geqq 2$ and $q = mk+1$ then for properly chosen $a \in GF(q)$ the polynomial

$$f(x) = x^{m+1} + ax$$

is a permutation polynomial for $GF(q)$, provided q exceeds a certain bound N_k. The proof is similar to the proof of Theorem 7 but requires an estimate for the number of solutions of certain systems of equations in a finite field.

Theorem 8. *Let $f(x)$ satisfy the hypotheses of the last theorem. Then $f(x)$ is not a permutation polynomial for any $GF(q^r)$ with $r > 1$.*

If r is even we have

$$q^r \equiv 1 \pmod{m-1}$$

and the stated result follows immediately. We therefore assume that $r = 2s+1$. Put

$$q^{2s+1} = k(m+1) + n; \tag{4. 8}$$

since
$$q^{2s+1} \equiv -1 \pmod{m+1},$$

it is clear that an integer k can be found for which (4. 8) is satisfied. We shall consider

$$(f(x))^{k+m-1} = (x^{m+1}+ax)^{k+m-1} = \sum_{j=0}^{k+m-1} \binom{k+m-1}{j} a^j x^{(m+1)(k+m-j-1)+j}. \tag{4. 9}$$

Since $s \geqq 1$ it follows easily that

$$q^{2s+1} \leqq (m+1)(k+m-1) < 2(q^{2s+1}-1).$$

Thus reducing (4.9) (mod $x^{q^{2s+1}}-x$) the only term that need be considered is the one corresponding to $j = m-1$, that is

$$\binom{k+m-1}{m-1} a^{m-1} x^{q^{2s+1}} - 1. \tag{4.10}$$

Now it follows form (4. 8) that $k(m+1) \equiv m+1 \pmod q$. Since $q = 2(m+1)-1$ we have $(m+1, q) = 1$ and therefore $k \equiv 1 \pmod q$.

We shall require the following known property of binomial coefficients. Let

$$r = r_0 + r_1 p + r_2 p^2 + \dots \qquad (0 \leqq r_j < p),$$
$$s = s_0 + s_1 p + s_2 p^2 + \dots \qquad (0 \leqq s_j < p),$$

where p is a prime. Then

$$\binom{r}{s} \sim \binom{r_0}{s_0} \binom{r_1}{s_1} \binom{r_2}{s_2} \dots \pmod p. \tag{4.11}$$

In particular if $r = ap^n + b$ $(0 \leqq b < p^n)$ $s = cp^n + d$ $(0 \leqq d < p^n)$, then (4. 11) implies

$$\binom{r}{s} \equiv \binom{a}{c} \binom{b}{d} \pmod p. \tag{4.12}$$

Returning to (4. 10) we put $k = tp^n + 1$, where $q = p^n$. Since $m < p^n$ it follows form (4. 12) that

$$\binom{k+m-1}{m-1} = \binom{tp^n+m}{m-1} \equiv m \not\equiv 0 \pmod p.$$

Thus (4. 10) is not zero and therefore $f(x)$ is not a permutation polynomial for $GF(q^{2s+1})$.

5. Let r be a fixed integer $\geqq 1$. We now briefly consider the set of transformations

$$y_i = f_i(x_1, \dots, x_r) \qquad (i = 1, \dots, r) \tag{5.1}$$

that possess an inverse of the same general form; the coefficients of the polynomial f_i lie in the fixed field $GF(q)$. The totality of all transformations (5. 1) constitute a group $\Gamma_r(q)$ isomorphic with the symmetric group on q^r letters. For some properties of polynomials relative to $\Gamma_r(q)$ see [1].

We can set up a correspondence between $\Gamma_r(q)$ and $\Gamma_1(q^r)$ in the following way. Let $\omega_1, \dots, \omega_r$ denote a basis of $GF(q^r)$ relative to $GF(q)$ and put

$$u = x_1\omega_1 + \dots + x_r\omega_r, \quad v = y_1\omega_1 + \dots + y_r w_r. \tag{5.2}$$

By means of (5. 1) every n-tuple $(x_1, \dots, x_n)$ of the $GF(q)$ is carried into the n-tuple $(y_1, \dots, y_n)$. By means of (5. 2) to the n-tuple $(x_1, \dots, x_n)$ corresponds the number u of $GF(q^r)$ and to the n-tuple $(y_1, \dots, y_n)$ corresponds the number v of $GF(q^r)$. Clearly the correspondence between u and v is one to one. We may accordingly write

$$v = f(u), \tag{5. 3}$$

where $f(u)$ is a permutation polynomial for $GF(q^r)$. Conversely if (5. 3) is given it is evident that (5. 1) is uniquely determined. We have therefore established a one to one correspondence between (5. 1) and (5. 3). This correspondence is evidently an isomorphism.

We may state

Theorem 9. *To every invertible transformation* (5. 1) *there corresponds the permutation* (5. 3) *and conversely. This correspondence induces an isomorphism between* $\Gamma_r(q)$ *and* $\Gamma_1(q^r)$.

If ξ denotes the column vector $(x_1, \ldots, x_r)$ and η the column vector $(y_1, \ldots, y_r)$, (5. 1) can be written compactly in the form

$$\eta = \varphi(\xi), \tag{5. 4}$$

where φ is a vector function of the vector ξ; $\varphi = (f_1, \ldots, f_r)$.

We shall now define two special transformations (5. 4), first the linear transformation

$$\eta = A\xi + \beta, \tag{5. 5}$$

where A is a non-singular matrix of order r and β is a column vector; the elements of both A and β are in $GF(q)$. In the second place corresponding to the transformation

$$u \to u^{q^r-2}$$

we define an involution

$$\eta = \xi^\sigma = (x_1^\sigma, \ldots, x_r^\sigma) \tag{5. 6}$$

by means of

$$(x_1\omega_1 + \ldots + x_r\omega_r)^{q^r-2} = x_1^\sigma\omega_1 + \ldots + x_r^\sigma\omega_r. \tag{5. 7}$$

Then we have the following

Theorem 10. *Every transformation of the group* $\Gamma_r(q)$ *can be generated by the special transformation* (5. 5) *and* (5. 6).

It is evidently not necessary to use all the transformations (5. 5). It would suffice to restrict A to a certain cyclic subgroup of nonsingular matrices of order $q^r - 1$. We shall however not take the space to state a stronger version of Theorem 10.

We remark that the involution (5. 6) is not uniquely determined but is dependent upon the choice of basis $\omega_1, \ldots, \omega_r$. If we make a change of basis:

$$\omega' = Cu, \tag{5. 8}$$

where w is the column vector $(\omega_1, \ldots, \omega_r)$ and C is a non-singular matrix with elements in $GF(q)$, then (5. 7) becomes

$$(x_1'\omega_1' + \ldots + x_r'\omega_r')^{q^r-2} = x_1'^\tau\omega_1' + \ldots + x_r'^\tau\omega_r', \tag{5. 9}$$

where τ is the involution corresponding to the ω_i' and

$$\xi = C^t \xi', \quad \xi' = (x_1', \ldots, x_r');$$

C^t is the transpose of C. Comparing (5. 9) with (5. 8) it is evident that

$$\xi'^{\tau} = (C^t)^{-1} (C^t \xi')^{\sigma}. \tag{5. 10}$$

This proves

Theorem 11. *Under the change of basis* (5. 8) *the involutions* σ, τ *corresponding to* ω_i, ω_i', *respectively, are related by means of* (5. 10).

The special transformation $(q > 2)$

$$y_1 = x_i^{q-2} \qquad (i = 1, \ldots, r) \tag{5. 11}$$

is an involution. However for $r > 1$ it cannot be identified with any of the involutions (5. 7). If we assume that (5. 11) can be defined by means of (5. 7) then it follows that

$$(x_1 \omega_1 + \ldots + x_r \omega_r)(x_1^{q-2} \omega_1 + \ldots + x_r^{q-2} \omega_r) = 1 \tag{5. 12}$$

for all $x_1, \ldots, x_r \in GF(q)$ except $(0, \ldots, 0)$. We may assume that $\omega^2 \neq 1$. Then if we take $x_1 = \ldots = x_{r-1} = 0$, $x_r = 1$, (5. 12) leads to a contradiction.

When $q = 3$ the transformation (5. 11) reduces to the identity; for $r > 1$ the transformations (5. 5) generate a proper subgroup of $\Gamma_r(q)$. It would be of interest to identify the group generated by (5. 5) and (5. 11) when $q > 3$ and $r > 1$.

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Compactification, Baire functions, and Daniell integration

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Introduction

The purpose of this paper is to expose precise relations within and between the domains of compactification, Baire functions, and Daniell integration. The principal method exploited to this end is a new topology, called the ι-topology, which we have developed on an earlier occasion [5], which when introduced into any compact space is finer than the compact or β-topology. Our interest in compactification centers upon the more delicate notions of pseudo-, countable, and real-compactifications. It seems fair to say that up to the present time these have not always been defined and approached in such a way as to make their theories completely transparent. In terms of the ι-topology, all the essential results are quite elementary.

In order to carry through this study, it is necessary to construct the standard compactification. We abandon the traditional notion of compactifying a completely regular space with respect to the set of *all* bounded continous functions in favor of the much more precise setting made available by contemporary topological algebra of compactifying an arbitrary set of points with respect to a Banach algebra of functions defined over it. The compactification is of course the structure space. This procedure gives *all* compactifications, not merely the Stone—Čech compactification, a very real advantage related to the fact that the notion of an algebra is finer than that of a topology (distinct algebras correspond to the same topology). The standard compactification is carried through using the notion of maximal positive cones of functions. This method is essentially equivalent to that of maximal convex ideals. It may be pointed out in passing that the method of filters of zero-sets has the very real disadvantage that a most interesting class of singular functions, precisely those defining the pseudocompactification have no zeros and therefore are automatically excluded from any filter procedure.

It should be mentioned in this connection that the question of obtaining all compactifications of a given completely regular space has been studied in detail by YU. SMIRNOV who based his results on the theory of proximities. We shall not elaborate on the rapport between these two formulations of the problem.

The compactification of a set $\mathscr{E}$ with respect to a Banach algebra of functions $\mathbf{B}$ is obtained by adjoining to $\mathscr{E}$ all maximal positive cones. These are of two types,

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the strong which contain functions (singular!) with empty zero-sets and the residual or weak cones. The realcompactification is obtained by adjoining to the base space $\mathscr{E}$ the set of points consisting of the weak cones. If one adjoins the strong cones, one obtains a pseudocompactification (equivalent to countable compactification for zero-sets). Adjoing to $\mathscr{E}$ both types, one obtains the compact space $\mathscr{E}^\frown$ and the algebra $\mathbf{B}^\frown$ of all continuous functions over $\mathscr{E}^\frown$. The β-topology over $\mathscr{E}^\frown$ is the weak topology induced by the functions of $\mathbf{B}^\frown$.

Starting with the functions in $\mathbf{B}^\frown$, one constructs the algebra $\mathbf{I}$ of all bounded Baire functions over $\mathscr{E}^\frown$. The ι-topology is the weak topology induced over $\mathscr{E}$ by the functions of $\mathbf{I}$. It is central to all the problems we consider. For example, the realcompactification of $\mathscr{E}$ introduced by E. HEWITT is the ι-closure in $\mathscr{E}^\frown$ of $\mathscr{E}$. Or again, $\mathscr{E}$ is pseudocompact if and only if it is ι-dense in $\mathscr{E}^\frown$. These topological facts explain why there is a unique realcompactification but no unique pseudocompactification. The latter is characterized in nine distinct ways.

While on this subject one or two remarks may be made. The points corresponding to weak maximal cones are topologically very complicated to describe and for all purposes of denumerable analysis seem superfluous. Thus it may be that for a large class of problems pseudocompactness rather than compactness is desirable. For another thing, it would seem that the common distinction in the literature of two algebras over $\mathscr{E}$, that of the bounded and that of the unbounded functions, seems largely unnecessary. A space admits an unbounded continuous function if and only if it admits a bounded singular function without zeros.

Turning to integration, the nexus of questions concerning Daniell integrals can be elucidated in the frame of this investigation. Necessary and sufficient conditions are given on the measure distribution over $\mathscr{E}^\frown$ of a linear functional F in order that F be a Daniell integral or an anti-integral over $\mathscr{E}$. Finally, it is shown that the space $\mathscr{E}^\frown$ is realcompact with respect to the algebra $\mathbf{I}$.

For general information concerning rings of continuous functions, the reader is referred to the comprehensive work of GILLMAN and JERISON [1]. The author wishes to express his thanks to LEONARD GILLMAN and to HING TONG for reading the manuscript carefully and making many suggestions. In particular, TONG has pointed out that the principal purpose of the algebraic device of a cone $\mathfrak{C}$ is to introduce the class of $\mathscr{E}$-vicinities associated with it. This class contains more information than the standard filters. Once the family of these classes is obtained the proofs proceed essentially along topological lines.

1. Notations

We introduce the fundamental structures to be considered throughout this paper. We consider an arbitrary set $\mathscr{E}$ of points $x, y, z, \ldots$. Real valued functions on $\mathscr{E}$ will be represented by $f, g, h, \ldots$; thus $f(x)$ is real for each $x \in \mathscr{E}$. We consider, in particular, a Banach algebra $\mathbf{B}$ of bounded functions of this type. Note that $\mathbf{B}$ is commutative. The norm in $\mathbf{B}$ is the uniform one: $\|f\| = \sup_{x\in\mathscr{E}} |f(x)|$. The set $\mathbf{B}$ is assumed to contain the constants; the constant function "one" is indicated by $e: e(x) = 1$, $x \in \mathscr{E}$. Real numbers will frequently be represented by $\alpha, \beta, \lambda, \mu, \ldots$.

We shall suppose that **B** distinguishes points of $\mathscr{E}$: given $x, y\in\mathscr{E}$, $x\neq y$, there exists $f\in\mathbf{B}$ such that $f(x)\neq f(y)$.

The algebra **B** is, as is well known, a lattice. The lattice operations are indicated by $f\vee g$ and $f\wedge g$. Also, $f^+=f\vee 0$; $f^-=(-f)\vee 0$; and $|f|=f^++f^-$. If $f\in\mathbf{B}$, we say f is *singular* in case there exists no g in **B** such that $f\cdot g=e$. As usual, $f\geqq 0$ if and only if $f=|f|$. If f is not singular, it is called *regular*.

We shall use frequently the following fact: *Let $f\in\mathbf{B}$ and suppose $f\geqq 0$. Then f is singular if and only if for each $\varepsilon>0$ there exists an $x\in\mathscr{E}$ such that $f(x)<\varepsilon$.* We prove the non-trivial portion of this statement. Suppose f is bounded away from zero. To simplify the computation, assume also that $\|f\|=1$. Thus for $x\in\mathscr{E}$, $0\leqq 1-f(x)\leqq 1-\varepsilon$ for some ε, $0<\varepsilon<1$, and $\|e-f\|<1$. Thus $f^{-1}=\sum_{n=0}^{\infty}(e-f)^n$.

We shall write $f\rhd 0$ to indicate that $f\geqq 0$, that f is singular, and that for each $x\in\mathscr{E}, f(x)>0$. For any $f\in\mathbf{B}$, the set of zeros of f, $\{x: x\in\mathscr{E}, f(x)=0\}$, will be denoted by $\mathfrak{Z}(f)$. Thus $f\rhd 0$ implies $\mathfrak{Z}(f)=\emptyset$. We shall refer to $\mathfrak{Z}(f)$ as the zero set of f.

An elementary but critically important fact about zero-sets is: *The intersection of denumerably many zero-sets is a zero-set.* Proof: Let $\{f_n\}$ be a sequence in **B**. Replacing f_n by $g_n=|f_n|\wedge e$, we see that $\mathfrak{Z}(f_n)=\mathfrak{Z}(g_n)$ and that $\|g_n\|\leqq 1$.

The function $f=\sum_{n=1}^{\infty}2^{-n}g_n$ is clearly in **B** and $\mathfrak{Z}(f)=\bigcap_{n=1}^{\infty}\mathfrak{Z}(f_n)$.

2. Cones of functions

The fundamental idea basic to our further discussion is that of cone of positive functions to which, for short, we shall refer as a positive cone. *A non-empty set $\mathfrak{C}$ of functions f in* **B** *is said to form a positive cone* (p. c) if: (1) $f\in\mathfrak{C}$ *implies* $f\geqq 0$; (2) $f\in\mathfrak{C}$ *implies f is singular*; (3) $f, g\in\mathfrak{C}$ imply $\alpha f+\beta g\in\mathfrak{C}$ for $\alpha>0$, $\beta>0$. (Actually, positive homogeneity is not necessary in the definition; it is automatic for maximal positive cones.)

The totality of positive cones may be partially ordered by set inclusion. By ZORN's lemma, there exists a maximal positive cone (m. p. c) containing any given positive cone. By a *weak maximal positive cone* $\mathfrak{C}$ (w. m. p. c) is meant a m. p. c. such that $f\in\mathfrak{C}$ implies that there exists $x\in\mathscr{E}$ such that $f(x)=0$. In other words $\mathfrak{C}$ is a w. m. p. c. if and only if $\mathfrak{C}$ is a m. p. c. and $\mathfrak{C}$ contains no $f\rhd 0$. By a strong maximal positive cone $\mathfrak{C}$ (s. m. p. c.) is meant a m. p. c. which is not weak, hence one which contains at least one f for which $f\rhd 0$.

A p. c. $\mathfrak{C}$ is called a very strong positive cone (v. s. p. c) providing $f\in\mathfrak{C}$ implies $f\rhd 0$. By ZORN's lemma, there exists a maximal very strong positive cone (m. v. s. p. c.) containing any given v. s. p. c. Note that if $\mathfrak{C}$ is a m. v. s. p. c. then $\mathfrak{C}$ is not a m. p. c. since $0\notin\mathfrak{C}$ whereas 0 is in every m. p. c.

Let $\mathfrak{C}$ be a m. p. c. Suppose $f\in\mathfrak{C}$ and that g in **B** is such that $0\leqq g\leqq f$. Then $g\in\mathfrak{C}$. This may be seen from fact that $\{\alpha g+h: \alpha\geqq 0, h\in\mathfrak{C}\}$ is a p. c. including $\mathfrak{C}$. Since $\mathfrak{C}$ is maximal, $g\in\mathfrak{C}$. In particular, if $\mathfrak{C}$ is a m. p. c., and if we have $f\in\mathfrak{C}$, $g\in\mathbf{B}$, $g\geqq 0$, then $f+g\in\mathfrak{C}$ implies $g\in\mathfrak{C}$.

Suppose $\mathfrak{C}$ *is a* m. p. c. *Let* $\{f_n\}$ *be a sequence such that* $f_n \in \mathfrak{C}$ *and* $f = \sum_{n=1}^{\infty} f_n$ *converges uniformly. Then* $f \in \mathfrak{C}$. The proof follows. Note that $f \geqq 0$; also, since $f_1 + \dots + f_n$ (which is in $\mathfrak{C}$) is singular and by the uniform convergence of the above series, f is singular. The set $\{h: h = \alpha f + g, \alpha \geqq 0, g \in \mathfrak{C}\}$ is clearly a p. c. containing $\mathfrak{C}$. Since $\mathfrak{C}$ is maximal, $f \in \mathfrak{C}$.

Let x_0 be a fixed point in $\mathscr{E}$ and let $\mathfrak{C} = \{f: f \geqq 0, f(x_0) = 0\}$. Then $\mathfrak{C}$ is evidently a p. c. It is also a m. p. c. (and hence a w. m. p. c) as we shall show. For if $\mathfrak{C}$ were not maximal, it would be contained properly in a m. p. c. $\mathfrak{C}'$ and hence there would exist in $\mathfrak{C}'$ a function g such that $g(x_0) = 1$. Since $\mathfrak{C}'$ is maximal $e \wedge g \in \mathfrak{C}'$. Writing $h = e - e \wedge g$ we see that $h \geqq 0$ and that $h(x_0) = 0$ and hence $h \in \mathfrak{C}$. Thus $h + e \wedge g = e$ is in $\mathfrak{C}'$. This is impossible since e is regular. Thus $\mathfrak{C}$ is maximal.

A m. p. c. such as $\mathfrak{C}$ in the preceding paragraph is called a *fixed* m. p. c. All other m. p. c. are *free*. From now on, unless the contrary is expressly indicated, the phrase m. p. c. will always refer to a free m. p. c.

Let $\mathfrak{W}$ denote the class of w. m. p. c.; let $\mathfrak{S}$ denote the class of s. m. p. c.; let $\mathfrak{VS}$ denote the class of m. v. s. p. c. Note that by definition, $\mathfrak{W} \cap \mathfrak{S} = \emptyset$. We shall write also, abusing somewhat notational niceties, $\mathfrak{S} \cap \mathscr{E} = \emptyset$ and $\mathfrak{W} \cap \mathscr{E} = \emptyset$. We shall establish a $1-1$ correspondence between $\mathfrak{S}$ and $\mathfrak{VS}$.

Theorem 1. *Let* $\mathfrak{C} \in \mathfrak{S}$ *and let* $\mathfrak{C}' = \{f: f \in \mathfrak{C}, f \rhd 0\}$. *Then* $\mathfrak{C}'$ *is a* m. v. s. p. c. *The mapping* $\Phi: \mathfrak{S} \to \mathfrak{VS}$ *defined by* $\Phi(\mathfrak{C}) = \mathfrak{C}'$ *is a bijection.*

Proof. We show first that $\mathfrak{C}'$ is a m. v. s. p. c. If $f, g \in \mathfrak{C}, f \rhd 0$ and $g \rhd 0$ then $f + g \rhd 0$ (note that $f + g \in \mathfrak{C}$ and hence is singular) and $\alpha f \rhd 0$ for $\alpha > 0$. Thus $\mathfrak{C}'$ is a v. s. p. c. It is easy to see that $\mathfrak{C}$ may be reconstructed from $\mathfrak{C}'$ as follows: For $f \in \mathfrak{C}'$ consider the set $\mathfrak{M}_f$ of all $g \geqq 0$ such that $g \leqq f$. Then $\mathfrak{C} = \bigcup_{f \in \mathfrak{C}'} \mathfrak{M}_f$. (Note that if $g \in \mathfrak{C}, f \in \mathfrak{C}'$ then $g + f \in \mathfrak{C}'$ and $g \leqq g + f$.) This argument shows that the mapping Φ is an injection (one-to-one) into the set of v. s. p. c.

Let $\mathfrak{D}$ be any m. v. s. p. c. which contains $\mathfrak{C}'$. Let $\mathfrak{E}$ be any m. p. c. which contains $\mathfrak{D}$. Then $\Phi(\mathfrak{E}) \supset \mathfrak{D}$ and since $\mathfrak{D}$ is a m. v. s. p. c., $\Phi(\mathfrak{E}) = \mathfrak{D}$. By the preceding paragraph, since $\mathfrak{D} \supset \mathfrak{C}'$, $\mathfrak{E} \supset \mathfrak{C}$. Since $\mathfrak{C}$ is maximal $\mathfrak{E} = \mathfrak{C}$ and $\mathfrak{D} = \mathfrak{C}'$. This shows that the range of Φ lies in $\mathfrak{VS}$; a slight variation to the above argument shows that Φ is in fact a bijection on $\mathfrak{VS}$.

By virtue of this theorem we may identify the sets $\mathfrak{S}$ and $\mathfrak{VS}$.

3. Adjunction of ideal points

In this section we adjoin "ideal" points to $\mathscr{E}$ produce various compactifications. Also we show how to extend any function f in **B** to these new points. The adjunction of points is a simple matter. Each m. p. c. $\mathfrak{C}$ will be called a point and will be denoted, according to need, by $\mathfrak{C}$, $x_{\mathfrak{C}}$ or x. The extension of the functions to these new points requires some investigation. In the meanwhile we may introduce the following

Definition 1. *The set* $\mathscr{E} \cup \mathfrak{W}$ *will be denoted by* $\mathscr{E}'$. *The set* $\mathscr{E} \cup \mathfrak{S}$ *will be denoted by* $\mathscr{E}^{\backprime}$. *The set* $\mathscr{E} \cup \mathfrak{W} \cup \mathfrak{S}$ *will be denoted by* $\mathscr{E}^{\frown}$.

Note that $\mathscr{E}' \cap \mathscr{E}^{\backprime} = \mathscr{E}$; $\mathscr{E}' \cup \mathscr{E}^{\backprime} = \mathscr{E}^{\frown}$.

Our first step toward extending the functions of $\mathbf{B}$ to the new points in $\mathscr{E}^\frown$ is to introduce a notion of $\mathscr{E}$-vicinity of an ideal point. Let $\mathfrak{S}$ be any m. p. c. and let $f \in \mathfrak{S}$, $\varepsilon > 0$. Then the set $\mathfrak{U}(f, \varepsilon) = \{x: f(x) < \varepsilon, x \in \mathscr{E}\}$ is called an *$\mathscr{E}$-vicinity* of $\mathfrak{S}$. Note that since f is singular, $\mathfrak{U}(f, \varepsilon)$ is not empty. If $\mathfrak{U}_1 = \mathfrak{U}(f_1, \varepsilon_1)$ and $\mathfrak{U}_2 = \mathfrak{U}(f_2, \varepsilon_2)$ are given, then, setting $\varepsilon = \inf(\varepsilon_1, \varepsilon_2)$, $f = f_1 + f_2$, and $\mathfrak{U} = \mathfrak{U}(f, \varepsilon)$, we have $\mathfrak{U} \subset \mathfrak{U}_1 \cap \mathfrak{U}_2$. Thus the intersection of a finite number of $\mathscr{E}$-vicinities contains an $\mathscr{E}$-vicinity.

Theorem 2. *Let $\mathfrak{S}$ be any* m. p. c. *Let $f \in$* B. *Then there exists a unique real number $\lambda (\lambda = \lambda(f, \mathfrak{S}))$ with the property: For every $\varepsilon > 0$ there exists an $\mathscr{E}$-vicinity $\mathfrak{U}$ such that $x \in \mathfrak{U}$ implies $|f(x) - \lambda| < \varepsilon$. The function $|f|$ is in $\mathfrak{S}$ if and only if $\lambda = 0$.*

Proof. For an arbitrary $\delta > 0$ and g in $\mathfrak{S}$, let $\mathfrak{U} = \mathfrak{U}(g, \delta)$. For $f \in \mathbf{B}$, let $M_{\mathfrak{U}} = \{\alpha: x \in \mathfrak{U}$ and $f(x) = \alpha\}$. The sets $M_{\mathfrak{U}}$ have the finite intersection property. Since for each $\mathfrak{U}$, $M_{\mathfrak{U}}$ lies in the closed interval $\{\alpha: |\alpha| \leqq \|f\|\}$, there is a point λ common to the closure of all sets $M_{\mathfrak{U}}$. The function $|f - \lambda e|$ is singular and the above argument demonstrates the singularity of $|f - \lambda e| + g$ where $g \in \mathfrak{S}$. Since $\mathfrak{S}$ is maximal, $|f - \lambda e| \in \mathfrak{S}$. If λ and μ belong to the closure of all sets $M_{\mathfrak{U}}$ then since $|\lambda - \mu| e \leqq |f - \lambda e| + |f - \mu e| \in \mathfrak{S}$, it is clear that $\lambda = \mu$. Now let $\varepsilon > 0$ be given. Let $\mathfrak{U} = \mathfrak{U}(|f - \lambda e|, \varepsilon)$. Then $x \in \mathfrak{U}$ implies $|f(x) - \lambda| < \varepsilon$.

Suppose $f \in \mathfrak{S}$ and let the number associated to f be λ. Then the previous paragraph shows that $|f - \lambda e| \in \mathfrak{S}$. Since $|f - 0e| \in \mathfrak{S}$ and since the number λ is unique, we have $\lambda = 0$. Now suppose that $f \in \mathbf{B}$ and that for the associated number λ, we have $\lambda = 0$. Then, as above, $|f - \lambda e| \in \mathfrak{S}$, that is, $|f| \in \mathfrak{S}$. This completes the proof.

Theorem 3. *Let $\mathfrak{S}$ be a* m. p. c. *and let $\Psi = \Psi_{\mathfrak{S}}$ be the mapping: $f \to \lambda$ of the preceding theorem. Then $\Psi_{\mathfrak{S}}$ is an algebra and lattice homomorphism. Thus if $\psi_{\mathfrak{S}} f = \lambda$ and $\psi_{\mathfrak{S}} g = \mu$, then*

(1) $\psi_{\mathfrak{S}} \alpha f = \alpha\lambda$;

(2) $\psi_{\mathfrak{S}}(f+g) = \lambda + \mu$;

(3) $\psi_{\mathfrak{S}}(f \cdot g) = \lambda \cdot \mu$;

(4) $\psi_{\mathfrak{S}}|f| = |\lambda|$.

If in an $\mathscr{E}$-vicinity $\mathfrak{U}$, $f(x)$ is close to λ and if in $\mathfrak{V}$, $g(x)$ is close to μ, then in $\mathfrak{U} \cap \mathfrak{V}$, $f + g$ is close to $\lambda + \mu$ and $f \cdot g$ is close to $\lambda \cdot \mu$. Similarly for the remaining cases.

Given $f \in \mathbf{B}$, we shall extend it to be a function $f^\frown$ over $\mathscr{E}^\frown$ in the following manner: if $x \in \mathfrak{W} \cup \mathfrak{S}$, that is, if $x \in \mathscr{E}^\frown - \mathscr{E}$, we set $f^\frown(x) = \lambda$, where the number λ is defined in the theorem 2. If $x \in \mathscr{E}$, we set $f^\frown(x) = f(x)$. By virtue of theorem 3, the algebraic and lattice-theoretic operations on the functions f commute with the extension operation "$^\frown$". That is, $(f+g)^\frown(x) = f^\frown(x) + g^\frown(x) = (f^\frown + g^\frown)(x)$, hence $(f+g)^\frown = f^\frown + g^\frown$; etc.

The totality of functions $f^\frown$ is called $\mathbf{B}^\frown$. It is clear from theorem 2 that $\sup_{x \in \mathscr{E}^\frown} |f^\frown(x)| = \sup_{x \in \mathscr{E}} |f(x)|$, hence we may introduce the supremum norm in $\mathbf{B}^\frown$ and we have $\|f^\frown\| = \|f\|$. This (and the preceding paragraph) shows that the mapping

$f \to f^\frown$ is an algebraic, lattice-theoretic, and metric isomorphism of $\mathbf{B}$ onto $\mathbf{B}^\frown$. In particular the spaces have isomorphic adjoint (or dual) spaces.

If $f^\frown \in \mathbf{B}^\frown$, and if $\mathscr{E}^\sim$ is a subset of $\mathscr{E}^\frown$ for which $\mathscr{E}^\sim \supset \mathscr{E}$, we shall denote by $f^\sim$ the restriction of $f^\frown$ to $\mathscr{E}^\sim$. The totality of functions $f^\sim$ will be denoted by $\mathbf{B}^\sim$. As before we see that the mapping $f \to f^\sim$ is an algebraic, lattice-theoretic, and metric isomorphism of $\mathbf{B}$ onto $\mathbf{B}^\sim$. If, in particular, $\mathscr{E}^\sim = \mathscr{E}'$ ($\mathscr{E}^\sim = \mathscr{E}'$), we shall be concerned with functions $f' \in \mathbf{B}'$ ($f' \in \mathbf{B}'$).

We shall introduce a topology in $\mathscr{E}^\frown$. Consider the family of zero-sets of the functions of $\mathbf{B}^\frown$: $\{\mathfrak{Z}(f^\frown): f^\frown \in \mathbf{B}^\frown\}$. Note that if $x \notin \mathfrak{Z}(f^\frown)$ and $x \notin \mathfrak{Z}(g^\frown)$, then $x \notin \mathfrak{Z}(f^\frown \cdot g^\frown)$. Thus the zero-sets of $\mathbf{B}^\frown$ are the base for the closed sets of a topology. It is this topology which is introduced into $\mathscr{E}^\frown$. It is easy to see that this topology is precisely the weak topology on $\mathscr{E}^\frown$; that is, it is the coarsest topology which renders continuous all the functions in $\mathbf{B}^\frown$. The topology thus introduced in $\mathscr{E}^\frown$ will be called the *β-topology.* The β-topology on $\mathscr{E}^\frown$ is a separeted or Hausdorff topology. To show this, it suffices to prove (since the β-topology is defined by means of the functions $f^\frown \in \mathbf{B}^\frown$) that $\mathbf{B}^\frown$ distinguish points in $\mathscr{E}^\frown$. If $x_0 \in \mathscr{E}$, it is easy to see that the totality of functions $f \geqq 0$ such that $f(x_0) = 0$ is a fixed m. p. c. If $x_0 \in \mathscr{E}^\frown - \mathscr{E}$ then by theorem 2, the totality of functions $f \geqq 0$ such that $f^\frown(x_0) = 0$ is a free m. p. c. Now, let $x, y \in \mathscr{E}^\frown$, $x \neq y$. Then if $x, y \in \mathscr{E}$, there exists by hypothesis an $f \in \mathbf{B}$ such that $f(x) \neq f(y)$. If $x \in \mathscr{E}$ and $y \in \mathscr{E}^\frown - \mathscr{E}$, then the two maximal positive cones in question cannot be the same since one is fixed and the other free. Thus, once more, there exists $f \in \mathbf{B}$ such that $f^\frown(x) \neq f^\frown(y)$. If $x, y \in \mathscr{E}^\frown - \mathscr{E}$, then $x \neq y$ implies that there is an $f \in \mathbf{B}$ such that $f^\frown(x) = 0$ (see theorem 2) and $f^\frown(y) \neq 0$. Thus in all cases, we see that the functions of $\mathbf{B}^\frown$ distinguish points and hence the β-topology on $\mathscr{E}$ is separated. One of our principal purposes will be to introduce later another characteristic topology in $\mathscr{E}^\frown$. In order to avoid confusion, we shall speak of sets as being β-open or β-closed rather than merely open or closed. If $\mathscr{E} \subset \mathscr{E}^\sim \subset \mathscr{E}^\frown$, the relative topology on $\mathscr{E}^\sim$ is the weak topology on $\mathscr{E}^\sim$ generated by the functions of $\mathbf{B}^\sim$.

Notice that the $\mathscr{E}$-vicinities of a m. p. c. $\mathfrak{C}$ are the traces on $\mathscr{E}$ of β-open neighborhoods of the point $x_{\mathfrak{C}}$ associated to $\mathfrak{C}$. This shows that $\mathscr{E}$ is β-dense in $\mathscr{E}^\frown$ since for each m. p. c. $\mathfrak{C}$, the $\mathscr{E}$-vicinities of $\mathfrak{C}$ are not empty.

4. Compactness properties

We establish briefly the β-compactness properties of $\mathscr{E}^\frown$.

Theorem 4. *The space $\mathscr{E}^\frown$ with the β-topology is compact. The algebra $\mathbf{B}^\frown$ is the algebra of all β-continuous functions on $\mathscr{E}^\frown$.*

Proof. We show that if $\mathfrak{F}$ is any family of closed sets having the finite intersection property, there exists a point $x \in \mathscr{E}^\frown$ such that x belongs to each $\mathscr{F}$ in $\mathfrak{F}$. First of all, we replace each $\mathscr{F}$ by a class of zero-sets whose intersection is $\mathscr{F}$. This gives a family $\mathfrak{F}'$ of zero-sets having the finite intersection property.

Let $\mathfrak{D}$ denote a set of functions $f \in \mathbf{B}$ such that (1) $f^\frown \geqq 0$; (2) for each $f \in \mathfrak{D}$, $\mathfrak{Z}(f^\frown)$ is a set in $\mathfrak{F}'$; (3) for each $\mathscr{F} \in \mathfrak{F}'$, there is an $f \in \mathfrak{D}$ such that $\mathfrak{Z}(f^\frown) = \mathscr{F}$. Let $\mathfrak{C}$ denote the set of all finite linear combinations $f = \alpha_1 f_1 + \ldots + \alpha_n f_n$, $f_i \in \mathfrak{D}$,

$\alpha_i \geqq 0$. Then $\mathfrak{C}$ is a p. c. since $f \geqq 0$ and since by the finite intersection property, f is singular. Now $\mathfrak{C}$ is included in a m. p. c. $\mathfrak{C}'$. Let $x_{\mathfrak{C}'}$ be the point in $\mathscr{E}^\frown$ associated with $\mathfrak{C}'$. Then $f \in \mathfrak{C} \Rightarrow f \in \mathfrak{C}' \Rightarrow f^\frown(x_{\mathfrak{C}'}) = 0$. Thus $x_{\mathfrak{C}'} \in \mathscr{F}$ for each $\mathscr{F} \in \mathfrak{F}$.

We have seen earlier that the β-topology on $\mathscr{E}^\frown$ is separated. Thus the above paragraph shows that $\mathscr{E}^\frown$ is β-compact. Now, note that $\mathbf{B}^\frown$ separates points of $\mathscr{E}^\frown$ and is uniformly closed. Thus by the Stone—Weierstrass theorem, $\mathbf{B}^\frown$ is the set of all β-continuous functions on $\mathscr{E}^\frown$.

Definition 2. *The space $\mathscr{E}^\sim \supset \mathscr{E}$ is β-countably compact if any countable covering of $\mathscr{E}^\sim$ by co-zero sets has a finite subcover.*

Note that the usual definition involves open sets and not co-zero sets.

Theorem 5. *The space $\mathscr{E}^\backprime$ is β-countably compact.*

We have seen in section 1 that the denumerable intersection of zero-sets is a zero-set. Let $\mathfrak{F}$ be any denumerable family of zero-sets in $\mathscr{E}^\backprime$, $\mathfrak{F} = \{\mathscr{F}_1, \mathscr{F}_2, \ldots\}$, which has the finite intersection property. Assume for a moment that $\mathscr{F} = \emptyset$ where $\mathscr{F} = \bigcap_{n=1}^{\infty} \mathscr{F}_n$. Then by a now standard construction there exists a positive singular function f such that $f \rhd 0$. Clearly, f is in some s. m. p. c. $\mathfrak{C}$ and hence $x_{\mathfrak{C}} \in \mathscr{E}^\backprime$ by definition. Also $f^\backprime(x_{\mathfrak{C}}) = 0$, a contradiction. Hence $\mathscr{F} \neq \emptyset$ and the theorem is proved.

Definition 3. *A space $\mathscr{E}^\sim \supset \mathscr{E}$ is said to be β-realcompact if and only if given any point $x \in \mathscr{E}^\frown - \mathscr{E}^\sim$, there exists a function $f \in \mathbf{B}$ such that $f^\sim \rhd 0$ and $f^\frown(x) = 0$.*

Thus, if $\mathscr{E}^\sim$ is β-realcompact, for $x \in \mathscr{E}^\frown - \mathscr{E}^\sim$, there exists a function which is β-continuous on $\mathscr{E}^\sim$ (of the form $f^{\sim -1}$) which cannot be extended continuously to any set containing x.

Theorem 6. *The space $\mathscr{E}'$ is β-realcompact. It is the smallest β-realcompact space containing $\mathscr{E}$.*

Proof. We have seen in § 2 that if $\mathfrak{C}$ is a m. p. c., if $f_n \in \mathfrak{C}$, $n = 1, 2, \ldots$, and if $f = \sum_{n=1}^{\infty} f_n$ converges uniformly, then $f \in \mathfrak{C}$. Moreover, if $\mathfrak{C}$ is a w. m. p. c. then $\mathscr{Z}(f) \neq \emptyset$. Since $\mathscr{Z}(f) = \bigcap_{n=1}^{\infty} \mathscr{Z}(f_n)$, this implies that for *any* sequence $\{f_n\}$ in a w. m. p. c., $\bigcap_{n=1}^{\infty} \mathscr{Z}(f_n) \neq \emptyset$.

We show that $\mathscr{E}'$ is β-realcompact. Let $x_0 \in \mathscr{E}^\frown - \mathscr{E}'$. Then x_0 arises from a s. m. p. c., hence there exists a function $f \in \mathrm{B}$ such that $f \rhd 0$ and $f^\frown(x_0) = 0$. It can easily be seen that $f' \rhd 0$, that is, that $x \in \mathscr{E}' - \mathscr{E}$ implies $f'(x) > 0$. For if $\mathfrak{C}_x$ denotes the w. m. p. c. corresponding to x, then $f'(x) = 0$ implies that $\mathscr{Z}(f) \neq \emptyset$. Thus $f \rhd 0$ implies $f' \rhd 0$.

We show now that if $\mathscr{E}^\sim$ is such that $\mathscr{E}^\sim \supset \mathscr{E}$ and if $\mathscr{E}^\sim$ is β-realcompact then $\mathscr{E}^\sim \supset \mathscr{E}'$. Since $\mathscr{E}^\sim$ is β-realcompact, there exists for each $x \in \mathscr{E}^\frown - \mathscr{E}^\sim$ an $f \in \mathrm{B}$ such that $f^\sim \rhd 0$ while $f^\frown(x) = 0$. Clearly, $f^\sim \rhd 0$ implies $f \rhd 0$; and the argument in the previous paragraph shows that $f \rhd 0$ implies $f' \rhd 0$. Invoking definition 3, we see that $\mathscr{E}^\sim \supset \mathscr{E}'$.

Theorem 7. *A space $\mathscr{E}$ is β-compact if and only if it is β-countably compact and β-realcompact.*

Proof: If $\mathscr{E}$ is β-compact, then all m. p. c. are fixed and hence $\mathscr{W}=\emptyset$ and $\mathscr{S}=\emptyset$. This implies that the sets $\mathscr{E}, \mathscr{E}', \mathscr{E}^{`}, \mathscr{E}^{\frown}$ are all identical. By theorem 5, $\mathscr{E}$ is β-countably compact. By theorem 6, $\mathscr{E}$ is β-realcompact.

Now suppose $\mathscr{E}$ is β-realcompact and β-countably compact. Then by theorem 6, $\mathscr{E}=\mathscr{E}'$, that is, $\mathscr{W}=\emptyset$. We have seen in the proof of theorem 5 that a space is β-countably compact if and only there does not exist a function $f\in\mathbf{B}$ such that $f\triangleright 0$; in other words, if and only if $\mathscr{S}=\emptyset$. Thus our hypotheses allow us to write $\mathscr{E}^{\frown}= =\mathscr{E}\cup\mathscr{W}\cup\mathscr{S}=\mathscr{E}$ and hence $\mathscr{E}$ is β-compact.

5. Baire functions and the ι-topology

In this section we introduce the ι-topology and indicate its relation to the Baire functions. Precise characterization of the sets $\mathscr{E}'$ and $\mathscr{E}^{`}$ in terms of the ι-topology are given.

The zero-sets in $\mathbf{B}$ (or in $\mathbf{B}^{\sim}$) have the property that the intersection of two zero-sets is a zero-set. This fact permits us to introduce:

Definition 4. *The ι-topology on $\mathscr{E}^{\frown}$ is that topology having for a base of its open sets the zero-sets of* $\mathbf{B}^{\frown}$.

The relative ι-topology in $\mathscr{E}^{\sim}$, $\mathscr{E}^{\frown}\supset\mathscr{E}^{\sim}\supset\mathscr{E}$, thus has as a base of its open sets the zero-sets of $\mathbf{B}^{\sim}$. The ι-topology was introduced in [5], p. 476. Proof of some of the statements of this section can be found in that work.

Definition 5. *The class of bounded Baire functions on $\mathscr{E}^{\frown}$ generated by $\mathbf{B}^{\frown}$ is the smallest class of bounded functions containing $\mathbf{B}^{\frown}$ and closed under the operation of taking pointwise limits. It will be denoted by* $\mathbf{I}$.

Thus, if $\{\varphi_n\}$ is a sequence of functions in $\mathbf{I}$ and if for each $x\in\mathscr{E}^{\frown}$, $\varphi_n(x)\uparrow\varphi(x)$ (or $\varphi_n(x)\downarrow\varphi(x)$) where φ is a bounded function, then $\varphi\in\mathbf{I}$. Notice that $\mathbf{I}$ is an algebra and a lattice and, if we set $\|\varphi\|=\sup_{x\in\mathscr{E}^{\frown}}|\varphi(x)|$, then $\mathbf{I}$ is a Banach space which contains $\mathbf{B}^{\frown}$.

Theorem 8. *The ι-topology on $\mathscr{E}^{\frown}$ is finer than the β-topology. It is the coarsest topology in which the functions of $\mathbf{I}$ are continuous. The denumerable intersection of ι-open sets is ι-open. The ι-topology is the coarsest topology containing the β-topology in which the set $\mathscr{E}^{\frown}$ is a P-space. The ι-topology is strictly finer than the β-topology if and only if $\mathscr{E}^{\frown}$ contains infinitely many points.*

Proof: If $\mathscr{M}$ is β-open and $x\in\mathscr{M}$, then, using the fact that $\mathscr{E}^{\frown}$ is β-compact and hence completely regular, and that each continuous function on $\mathscr{E}^{\frown}$ is in $\mathbf{B}^{\frown}$ (theorem 4), there exists a zero set $\mathscr{Z}(f^{\frown})$, $f\in\mathbf{B}$, containing x and contained in $\mathscr{M}$. Thus $\mathscr{M}$ is ι-open. Hence the ι-topology is finer than the β-topology. The proof of the second sentence of the theorem is to be found in [5], pp. 477—479. Since the denumerable intersection of zero-sets arising from $\mathbf{B}^{\frown}$ is a zero-set, the denumerable intersection of ι-open sets is ι-open. This shows also that $\mathscr{E}^{\frown}$ in the

ι-topology is a P-space — see [1], 4J (4), p. 63. If $\mathscr{E}^{\frown}$ is a P-space in some topology finer than the β-topology, this topology will contain denumerable intersections of β-open sets, hence will contain all zero-sets arising from $\mathbf{B}^{\frown}$. Thus the topology will contain the ι-topology. The proof of the last statement in the theorem follows after the next paragraph.

If $\varphi \in \mathbf{I}$, then the set $\mathscr{Z}(\varphi)$ will be called a *Baire* set. It is easy to verify that if $\psi \in \mathbf{I}$ and α is a real number, the set $\{x\colon \psi(x) \leqq \alpha\}$ is a Baire set. One may also see that the family of Baire sets is closed under complementation and the formation of denumerable unions and intersections. The class of Baire sets is the smallest class closed under these operations and containing the zero-sets arising from $\mathbf{B}^{\frown}$. Clearly all Baire sets are ι-open and ι-closed. Since the β-open cozero sets in $\mathscr{E}^{\frown}$ constitute a base for the ι-closed sets, and since these same sets are Baire sets, the ι-closure of any set is the intersection of all Baire sets which contain it.

If $\mathscr{E}$ is finite, so is $\mathscr{E}^{\frown}$ and hence $\mathbf{B}=\mathbf{I}$. Assume now that $\mathscr{E}$ has infinitely many points. We shall construct a Baire function φ on $\mathscr{E}^{\frown}$, hence a ι-continuous function, which assumes values arbitrarily close to 1 but which does not assume the value 1. Thus φ cannot be β-continuous since $\mathscr{E}^{\frown}$ is β-compact and since the range of each β-continuous function is a closed set. This will prove that the β-topology and the ι-topology are distinct, hence that the latter is strictly finer than the former.

We construct φ. If $x, y \in \mathscr{E}^{\frown}$, there exists a β-continuous function $f^{\frown}$ in $\mathbf{B}^{\frown}$ such that $f^{\frown}(x)=0$ and $f^{\frown}(y)\neq 0$. The sets $\mathfrak{M}_1 = Z(f^{\frown})$ and $\mathfrak{M}_2 = \mathscr{E}^{\frown} - \mathfrak{M}_1$ are Baire sets and at least one of them contains infinitely many points. Assume $\mathfrak{M}_2$ is infinite and let ψ_1 be the characteristic function of $\mathfrak{M}_1$. Note that ψ_1 is a Baire function. We now split $\mathfrak{M}_2$ into two non-empty Baire sets, of which the second $\mathfrak{M}_3$ is infinite and we let ψ_2 be the non-zero characteristic function of the first. The splitting is accomplished as follows: $\mathfrak{M}_2$, being a Baire set, is the union of zero sets from $\mathbf{B}^{\frown}$. Furthermore, if $x, y \in \mathfrak{M}_2$, $x \neq y$, there exists a zero set containing x and not y. Continuing in this way, we construct a sequence $\{\psi_n\}$ of non-zero Baire functions satisfying $\psi_n \cdot \psi_m = 0$ for $n \neq m$. Let $\varphi = \sum_{n=1}^{\infty} (1-2^{-n})\varphi_n$. Then φ has the properties asserted above. This concludes the proof.

Theorem 9. *The set $\mathscr{E}'$ is precisely the ι-closure of $\mathscr{E}$. Thus $\mathscr{E}'$ is the intersection of all Baire sets containing $\mathscr{E}$. Also, $\mathscr{W}$ is the set of ι-limit points of $\mathscr{E}$ lying in $\mathscr{E}^{\frown} - \mathscr{E}$. The operations of forming the β-realcompactification and of forming the ι-closure are identical.*

Proof. Suppose $x \in \mathscr{E}^{\frown} - \mathscr{E}$ and suppose that every zero-set containing x meets $\mathscr{E}$. Then $x \in \mathscr{W}$. Thus the ι-closure of $\mathscr{E}$ lies in $\mathscr{E}'$. If now $x \in \mathscr{W}$, each zero-set containing x intersects $\mathscr{E}$. This shows that the ι-closure of $\mathscr{E}$ is $\mathscr{E}'$. The last statement in the theorem follows from theorem 6.

Theorem 10. *The set $\mathscr{S}$ in $\mathscr{E}^{\frown}$ is the ι-interior of the complement of $\mathscr{E}$. The only Baire set containing $\mathscr{E}'$ is $\mathscr{E}^{\frown}$. Let $\varphi \in \mathbf{I}$ and let φ' represent the restriction of φ to $\mathscr{E}'$. Then the map: $\varphi \to \varphi'$ is an algebraic isomorphism from $\mathbf{I}$ onto the set of all Baire functions defined on $\mathscr{E}'$ and generated by the functions of $\mathbf{B}'$.*

Proof. The ι-interior of the complement of $\mathscr{E}$ equals the complement of the ι-closure of $\mathscr{E}$. The preceding theorem shows that this is $\mathscr{E}^{\frown} - \mathscr{E}' = \mathscr{S}$. Any Baire set containing $\mathscr{E}^{\backprime}$ contains $\mathscr{E}$ and hence contains $\mathscr{E}'$ by the preceding theorem. Thus the Baire set contains $\mathscr{E}' \cup \mathscr{E}^{\backprime} = \mathscr{E}^{\frown}$.

The map $\varphi \to \varphi^{\backprime}$ is obviously a homomorphism. Suppose $\varphi^{\backprime} = 0$. Thus $\mathscr{Z}(\varphi) \supset \mathscr{E}^{\backprime}$ and by the preceding paragraph, $\mathscr{Z}(\varphi) = \mathscr{E}^{\frown}$, that is, $\varphi = 0$. Thus the map is an isomorphism.

We show that the map is onto. If $f^{\backprime}$ is a Baire function of class zero on $\mathscr{E}^{\backprime}$ (hence continuous), there exists a unique function $f^{\frown} \in \mathbf{B}^{\frown}$ such that the restriction of $f^{\frown}$ to $\mathscr{E}^{\backprime}$ is $f^{\backprime}$. If $x_0 \in \mathscr{E}^{\frown} - \mathscr{E}^{\backprime}$, $x_0 \in \mathscr{W}$; hence for $f \in \mathbf{B}$, the set $\{x : f^{\frown}(x) = f^{\frown}(x_0)\}$ is a zero-set which intersects $\mathscr{E}$ and hence $\mathscr{E}^{\backprime}$. If $\{f_n\}$ is any sequence from $\mathbf{B}$, there is a zero-set $\mathscr{Z}$ intersecting $\mathscr{E}$ on which $f_n^{\frown}(x) = f_n^{\frown}(x_0)$, $n = 1, 2, \ldots$. Now let $\{f_n^{\backprime}\}$ denote a sequence of Baire functions of class 0 defined on $\mathscr{E}^{\backprime}$ and such that $f_n^{\backprime} \uparrow \psi$ pointwise. Then the associated sequence $\{f_n^{\frown}\}$ converges to a Baire function φ defined on $\mathscr{E}^{\frown}$ and $\varphi^{\backprime} = \psi$; in fact $\varphi^{\backprime}$ is constant on the zero-set $\mathscr{Z}$. Thus all Baire functions of order 1 on $\mathscr{E}^{\backprime}$ can be obtained by the mapping process of the theorem. The proof can now be extended to Baire functions of higher order, making use of the fact that for any sequence of such functions $\{\psi_n\}$, there exists a zero-set containing x_0 on which all functions are constant.

6. Pseudocompact spaces

We define the concept of pseudocompactness and set down equivalents for it.

Definition 6. *A space $\mathscr{E}$ is β-pseudocompact if every function $f \in \mathbf{B}$ such that $f(x) > 0$ for all $x \in \mathscr{E}$ is regular.*

In other words, there exists no function f such that $f \rhd 0$; hence $\mathscr{S} = \varnothing$.

Theorem 11. *The following statements are equivalent:*

(1) *$\mathscr{E} = \mathscr{E}^{\backprime}$ (or equivalently $\mathscr{E}$ is ι-dense in $\mathscr{E}^{\frown}$).*
(2) *$\mathscr{E}$ is β-pseudocompact.*
(3) *$\mathscr{E}$ is β-countably compact (for zero-sets).*
(4) *$\mathscr{E}$ intersects each non-empty zero set $\mathscr{Z}(f^{\frown})$, $f \in \mathbf{B}$.*
(5) *Each function in $\mathbf{B}$ assumes its $\mathscr{E}^{\frown}$ maximum and its $\mathscr{E}^{\frown}$ minimum in $\mathscr{E}$.*
(6) *For each $f \in \mathbf{B}$, the range of f is a closed set.*
(7) *If $f_n \in \mathbf{B}$, $n = 1, 2, \ldots$, and for each $x \in \mathscr{E}$, $f_n(x) \downarrow 0$, then $f_n \downarrow 0$ uniformly on $\mathscr{E}$.*
(8) *Each positive linear functional F over $\mathbf{B}$ is a positive Daniell integral.*
(9) *The map which restricts to $\mathscr{E}$ each Baire function on $\mathscr{E}^{\frown}$: $\varphi \to \varphi/\mathscr{E}$, is an isomorphism.*

Proof. It is trivial that (1) $\Rightarrow$ (2). The intersection of denumerably many zero-sets in $\mathscr{E}$ having the finite intersection property is empty if and only if there exists a positive singular f such that $f \rhd 0$. Thus (2) $\Rightarrow$ (3). Let $f \in \mathbf{B}$, let $\mathscr{Z}(f^{\frown}) \neq \varnothing$ and suppose $f \geqq 0$. Then $\mathscr{Z}(f) = \bigcap_{n=1}^{\infty} \{x : x \in \mathscr{E}, f(x) \leqq 1/n\}$. The zero-sets in this intersection are non-empty and have the finite intersection property. Thus (3) $\Rightarrow$(4). If $\max_{x \in \mathscr{E}^{\frown}} f^{\frown}(x) = \alpha$, then $\alpha e - f$ is singular. Thus (4) $\Rightarrow$ (5.) If β is a limit point of the range of f, then $|f - \beta e|$ is singular. Hence (5) $\Rightarrow$ (6).

Assume (6). Let $\{f_n\}$ be a sequence of functions such that $f_n(x)\downarrow 0$, $x\in\mathscr{E}$. Then the sequence of extended functions, $\{f_n^\frown\}$, converges monotonely for each $x\in\mathscr{E}^\frown$. In order to show uniform convergence it will be sufficient to show (since $\mathscr{E}^\frown$ is β-compact) that $x_0\in\mathscr{E}^\frown-\mathscr{E}$ implies $f_n^\frown(x_0)\downarrow 0$. This will be done by proving that there exists an y in $\mathscr{E}$ such that for each n, $f_n^\frown(x_0)=f_n(y)$. Write $\alpha_n=f_n^\frown(x_0)$. Then $|\alpha_n e-f_n|$ is singular. For a suitable sequence of constants $\{\beta_n\}$, $\beta_n>0$ (e. g., $\beta_n=2^{-n}\|\alpha_n e-f_n\|^{-1}$) $f=\sum_{n=1}^{\infty}\beta_n|\alpha_n e-f_n|$ converges uniformly hence $f\in\mathbf{B}$. Since $f^\frown(x_0)=0$, f is singular. By (6) there exists $y\in\mathscr{E}$ such that $f(y)=0$. This means that $f_n^\frown(x_0)=f_n(y)$ for all n and since by hypothesis $f_n(y)\downarrow 0$, we have $f_n^\frown(x_0)\downarrow 0$. Thus (6) $\Rightarrow$ (7).

A positive linear functional F is called a positive Daniell integral if it has the property: if $\{f_n\}$ is a sequence such that $f_n(x)\downarrow 0$ for all $x\in\mathscr{E}$, then $Ff_n\downarrow 0$. Now, let F be a positive linear functional over $\mathbf{B}$. Then F is bounded, with bound $\|F\|$. Let $\{f_n\}$ be any sequence from $\mathbf{B}$ such that $f_n(x)\downarrow 0$ for all x in $\mathscr{E}$. Then (7) implies that $f_n\downarrow 0$ uniformly, that is, that $\|f_n\|\downarrow 0$. We have therefore $Ff_n\leqq\|F\|\ \|f_n\|\downarrow 0$. Hence (7) $\Rightarrow$ (8).

Suppose the mapping $\varphi\to\varphi/\mathscr{E}$ is not an isomorphism. Then there exists a Baire function φ and a point $x_0\in\mathscr{E}^\frown-\mathscr{E}$ such that $\varphi(x_0)=1$ and $\varphi(x)=0$ for $x\in\mathscr{E}$. Clearly $x_0\in\mathbb{S}$. Hence there exists a positive singular function f such that $f\rhd 0$ and $f^\frown(x_0)=0$. Assume $f\leqq e$ and define $f_n = e\dot{-}(nf\wedge e)$. Then $x\in\mathscr{E}$ implies $f_n(x)\downarrow 0$. Let F be a positive linear functional defined by $Fg=g^\frown(x_0)$, $g\in\mathbf{B}$. Then $Ff_n=1$. Thus F is not a positive Daniell integral. This means that (8) $\Rightarrow$ (9).

If the mapping: $\varphi\to\varphi/\mathscr{E}$ is an isomorphism, the argument of the preceding paragraph shows that $\mathbb{S}=\emptyset$, hence that $\mathscr{E}=\mathscr{E}'$. Thus (9) $\Rightarrow$ (1). The theorem is proved.

For the space of all functions continuous and bounded on $\mathscr{E}$, the equivalence of (2), (5), (7), and (8) was shown by Glicksberg [2]; the relations (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (5) have been obtained by Hewitt [4]; the relation (2) $\Leftrightarrow$ (3) appears in Gillman and Jerison [1], p. 79.

7. Functionals and their associated measures

As indicated before, a positive Daniell integral is a positive linear functional F defined over a vector lattice of functions, which is continuous with respect to monotone pointwise convergence of functions. That is, $f_n(x)\downarrow 0$ for each x implies $Ff_n\downarrow 0$. If $\mathbf{B}$ is a Banach space of functions of the type we have been considering, each positive functional F is the sum of uniquely defined positive functionals G and H, $F=G+H$, where G is a positive Daniell integral and H will be called a positive anti-integral. For the complete discussion of this decomposition we refer the reader to [3]. (As one would expect, a Daniell integral is defined to be the difference of two positive Daniell integrals.) In the present work, there are two structures of interest with respect to Daniell integration: the algebra $\mathbf{B}$ of functions defined over $\mathscr{E}$ and the algebra $\mathbf{I}$ of Baire functions defined over $\mathscr{E}^\frown$. We systematize below the principal facts concerning the Daniell integrals over these algebras. For the algebra $\mathbf{B}^\frown$ of functions defined over $\mathscr{E}^\frown$, each linear functional is a Daniell integral. (We remind the reader that the set of all bounded linear functionals over B is denoted by $\mathbf{B}^*$.)

The mapping described in the next theorem is well known:

Theorem 12. *Let* $\mathbf{I}$ *denote the Banach algebra of bounded Baire functions over* $\mathscr{E}^\frown$. *Let* $\mathbf{M}^*$ *denote the closed linear manifold of Daniell integrals over* $\mathbf{I}$, *thus* $\mathbf{M}^* \subset \mathbf{I}^*$. *If* $F \in \mathbf{M}^*$, *let* $F/\mathbf{B}^\frown$ *denote the restriction of* F *to the functions of* $\mathbf{B}^\frown$. *Then the map* $F \to F/\mathbf{B}^\frown$ *is a vector-lattice isomorphism. The range of the map is all of* $(\mathbf{B}^\frown)^*$. *The inverse mapping is the (canonical) imbedding of* $(\mathbf{B}^\frown)^*$ *into* $\mathbf{I}^*$. *The latter mapping covers* $\mathbf{I}^*$ *if and only if* $\mathscr{E}$ *has a finite number of points.*

The fact that the map is a vector-lattice homomorphism is obvious. We show that the range is all of $\mathbf{B}^*$. (Since $\mathbf{B}^*$ and $(\mathbf{B}^\frown)^*$ are isometrically isomorphic, we use the simpler notation $\mathbf{B}^*$.) Let $\Lambda \in \mathbf{B}^*$. Then there is associated to Λ a measure λ defined for all Baire sets in $\mathscr{E}^\frown$ such that for $f^\frown \in \mathbf{B}^\frown$, $\Lambda f^\frown = \int f^\frown d\lambda$. Furthermore, the functional Λ may be extended to a larger class of functions — which includes the Baire functions. Denote the extended functional by F. Then F is defined over $\mathbf{I}$ and since, as an integral, it satisfies LEBESGUE's dominated convergence theorem, it is a Daniell integral. Clearly, $F/\mathbf{B}^\frown = \Lambda$. This proves the "onto" property. To show that the homomorphism is an isomorphism, assume that the Daniell integral $F \geqq 0$ in $\mathbf{I}^*$ is zero over $\mathbf{B}^\frown$. If φ is a Baire function of the first class and $f_n^\frown \uparrow \varphi$, $f_n^\frown \in \mathbf{B}^\frown$, then the Daniell property shows that $F\varphi = 0$; similarly for functions of higher order. Thus $F\varphi = 0$ for all Baire functions φ and the map of the theorem is an isomorphism.

If $\mathscr{E}$ is finite, then $\mathscr{E} = \mathscr{E}^\frown$ and $\mathbf{B} = \mathbf{B}^\frown = \mathbf{I}$. Thus the canonical mapping of $(\mathbf{B}^\frown)^*$ into $\mathbf{I}^*$ is surjective (onto). If $\mathscr{E}$ is infinite, then there exists a singular Baire function $\xi \geqq 0$ such that $\xi(x) > 0$ for each $x \in \mathscr{E}^\frown$; in other words $\xi \rhd 0$. For example, the function $\xi = e^\frown - \varphi$, where φ is the function constructed in the last paragraph of the proof of theorem 8, has this property. This implies that $\mathscr{E}^\frown$ is not ι-countably compact and hence by theorem 11 (8), there exists a linear functional over $\mathbf{I}$ which is not a Daniell integral, hence which does not belong to $\mathbf{M}^*$. This means that the canonical mapping of $(\mathbf{B}^\frown)^*$ into $\mathbf{I}^*$ is not surjective.

In the remainder of this section, F represents a linear functional over $\mathbf{B}$. Assume $F \geqq 0$. With respect to the base set $\mathscr{E}$, two extreme cases arise: F is a Daniell integral or F is an anti-integral. On the other hand, F considered as a linear functional over $\mathbf{B}^\frown$ has associated to it a countably additive measure defined for the Baire sets of $\mathscr{E}^\frown$. We shall say that the measure of F is concentrated on a set $\mathscr{A} \subset \mathscr{E}^\frown$ if the F-measure of any Baire set lying in $\mathscr{E}^\frown - \mathscr{A}$ is zero. The question arises to as the extent to which the F-measure of a Daniell integral is concentrated "close" to $\mathscr{E}$. The proper definition of "close" is given by the ι-topology. We shall show that the measure is concentrated on the ι-closure of $\mathscr{E}$, that is, on $\mathscr{E}^\iota$. Similarly, we shall show that F is an anti-integral if and only if its F-measure is concentrated in a Baire set whose complement contains $\mathscr{E}$.

Theorem 13. *Let* $F \geqq 0$ *be a linear functional over* $\mathbf{B}$. *Then* F *is a Daniell integral if and only if the* F*-measure of each Baire set in* $\mathfrak{S} = \mathscr{E}^\frown - \mathscr{E}'$ *is zero. Thus the measure of* F *is concentrated on each Baire set containing* $\mathscr{E}$.

Proof. Suppose that the F-measure of each Baire set in $\mathscr{E}^\frown - \mathscr{E}^\iota$ is zero. Let $f_n \in \mathbf{B}$, $f_n(x) \downarrow 0$, $x \in \mathscr{E}$. Then the set $\mathfrak{M} = \{x : f_n^\frown(x) \downarrow 0,\ x \in \mathscr{E}^\frown\}$ is a Baire set including

$\mathscr{E}$, hence including $\mathscr{E}'$. Its complement is a Baire set which lies in $\mathbb{S}$. Since the F-measure of this complement is zero, an application of LEBESGUE's convergence theorem and the fact that $\mathscr{E}^\frown = \mathfrak{M} \cup (\mathscr{E}^\frown - \mathfrak{M})$ shows that $Ff_n \downarrow 0$. Thus F is a Daniell integral.

Now, assume that the linear functional F is a Daniell integral. Let μ be the measure associated to F and let φ represent any bounded Baire function. Then F may be extended to all of $\mathbf{I}$ by means of the formula $F\varphi = \int \varphi d\mu$. According to the theory of the Daniell integral, if $\varphi_1(x) = \varphi_2(x)$ for each $x \in \mathscr{E}$, $F\varphi_1 = F\varphi_2$. Now let $\mathfrak{M}$ represent any Baire set lying in $\mathbb{S}$ and let $\chi_{\mathfrak{M}}$ be its characteristic function. Then $\chi_{\mathfrak{M}}(x) = 0$ for x in $\mathscr{E}$ and hence $F\chi_{\mathfrak{M}} = 0$. Thus $\mathfrak{M}$ has F-measure equal to zero. This proves the theorem.

Let $\mathbf{M}^*$ denote the closed linear manifold of Daniell integrals; and let $\mathbf{N}^*$ denote the closed linear manifold of anti-integrals. We have indicated that $\mathbf{B}^*$ is the direct sum of these two manifolds: $\mathbf{B}^* = \mathbf{M}^* \oplus \mathbf{N}^*$.

Theorem 14. *Let $F \geqq 0$ be in $\mathbf{B}^*$. Then F is an anti-integral — $F \in \mathbf{N}^*$ — if and only if there exists a Baire set $\mathfrak{M}$ lying in $\mathbb{S} = \mathscr{E}^\frown - \mathscr{E}'$ such that the measure of F is concentrated on $\mathfrak{M}$.*

Proof. Let $F \geqq 0$ be a linear functional over $\mathbf{B}$ whose measure is concentrated on a Baire set $\mathfrak{M} \subset \mathbb{S}$. Let $F = G + H$ where $G \in \mathbf{M}^*$ and $H \in \mathbf{N}^*$. Since $F \geqq G$, the measure of G is concentrated on $\mathfrak{M}$ and the G-measure of $\mathscr{E}^\frown - \mathfrak{M}$ is zero. Since G is a Daniell integral, the G-measure of $\mathfrak{M}$ is zero by theorem 13. Thus $G = 0$ and $F = H$, that is, $F \in \mathbf{N}^*$.

Assume next that $F \in \mathbf{N}^*$. Let us suppose first that every zero-set from $\mathbf{B}^\frown$ which lies in $\mathbb{S}$ is of F-measure zero. Let $f_n \in \mathbf{B}$, $n = 1, 2, \ldots$, and suppose $f_n(x) \downarrow 0$, $x \in \mathscr{E}$. Let $\mathscr{A}_r = \{x : f_n^\frown(x) \geqq 2^{-r} \text{ for all } n\}$. Then $\mathscr{A}_r$ is the intersection of denumerably many zero sets hence is a zero-set. Note that $\mathscr{A}_r \subset \mathscr{E}^\frown - \mathscr{E}'$, thus the F-measure of $\mathscr{A}_r$ is zero. Now, if we write $\mathscr{A} = \{x : f_n^\frown(x) \not\downarrow 0, x \in \mathscr{E}^\frown\}$, then $\mathscr{A} = \bigcup_{r=1}^{\infty} \mathscr{A}_r$. Thus $\mathscr{A}$ is a Baire set lying in $\mathscr{E}^\frown - \mathscr{E}$ and the F-measure of $\mathscr{A}$ is zero. By LEBESGUE's theorem, $Ff_n \downarrow 0$. Thus F is a Daniell integral. Since $F \in \mathbf{N}^*$, $F = 0$. We conclude that if $F \neq 0$ and $F \in \mathbf{N}^*$, there exist zero-sets in $\mathscr{E}^\frown - \mathscr{E}'$ of non-zero F-measure.

Suppose $\mathfrak{Z}$ is such a zero-set and suppose the F-measure of $\mathfrak{Z}$ is α. Then since $F > 0$, we have $0 < \alpha \leqq Fe$. Let λ_1 be the supremum of all values α so obtained. Let $\mathfrak{Z}_1$ be any zero-set whose F-measurable exceeds $\lambda_1/2$. Let χ_1 be the characteristic function of $\mathfrak{Z}_1$ and define F_1 by $F_1 f = F\chi_1 f$, $f \in \mathbf{B}$. (In this context, we write indiscriminately Ff and $Ff^\frown$.) Then $F_1 \in \mathbf{B}^*$ and since $F \geqq F_1 > 0$, $F_1 \in \mathbf{N}^*$. Note that the measure of F_1 is concentrated on $\mathfrak{Z}_1$ which is a Baire set whose complement contains $\mathscr{E}$. If $F - F_1 = 0$, the theorem is proved.

Suppose $F - F_1 \neq 0$; write $K_1 = F - F_1$. We have $K_1 > 0$ and $K_1 \in \mathbf{N}^*$. Note that the measure of K_1 is concentrated on the complement of $\mathfrak{Z}_1$. Let λ_2 be the supremum of all values α, where α represents the K_1-measure of an arbitrary zero-set lying in $\mathscr{E}^\frown - \mathscr{E}'$. Then $0 < \lambda_2 < 2^{-1}\lambda_1$. Let $\mathfrak{Z}_2$ be any zero-set in $\mathbb{S} - \mathfrak{Z}_1$ whose K_1-measure exceeds $\lambda_2/2$. Let χ_2 denote the characteristic function of $\mathfrak{Z}_2$ and define

F_2 by $F_2 f = K_1 \chi_2 f$, $f \in \mathbf{B}$. Then $F_2 \in \mathbf{B}^*$, and since $K_1 \geqq F_2 > 0$, $F_2 \in \mathbf{N}^*$. Write $K_2 = K_1 - F_2 = F - (F_1 + F_2)$. Note that the measure of K_2 is concentrated on the complement of $\mathfrak{Z}_1 \cup \mathfrak{Z}_2$. It is now clear how to construct the objects F_n, K_n, $\mathfrak{Z}_n$, χ_n, and λ_n, $n = 3, 4, \ldots$. If for any n, $K_n = 0$ the theorem is proved. Otherwise we have for all n, $K_n \in \mathbf{N}^*$, $K_n = F - (F_1 + \ldots + F_n)$ and the supremum λ_n of the K_n-measure of zero-sets in $\mathscr{E}^\frown - \mathscr{E}'$ satisfies $\lambda_n < 2^{-(n-1)}\lambda_1$, hence $\lambda_n \to 0$. Also the measure of K_n is concentrated on the complement of $\mathfrak{Z}_1 \cup \ldots \cup \mathfrak{Z}_n$. Let $K = F - \sum_{n=1}^{\infty} F_n$. Clearly $K \in \mathbf{N}^*$, $K_n \geqq K \geqq 0$. Since the K-measure of any zero set in $\mathscr{E}^\frown - \mathscr{E}'$ is inferior to λ_n, it is zero. Thus by the argument given earlier, $K = 0$ and $F = \sum_{n=1}^{\infty} F_n$. Note that the measure of F is concentrated on the set $\bigcup_{n=1}^{\infty} \mathfrak{Z}_n$ which is a Baire set and lies in $\mathscr{E}^\frown - \mathscr{E}'$. This concludes the proof.

9. Realcompactness and the Baire funtions

Given the set $\mathscr{E}^\frown$ and the algebra $\mathbf{I}$ of bounded Baire functions over $\mathscr{E}^\frown$, one may consider compactification questions of $\mathscr{E}^\frown$ with respect to $\mathbf{I}$ (just as at the outset we compactified $\mathscr{E}$ with respect to $\mathbf{B}$). Natural questions are: is $\mathscr{E}^\frown$ ι-realcompact, ι-countably compact, ι-compact. It has already been shown that $\mathscr{E}^\frown$ is not ι-countably compact, hence not ι-compact (theorem 12) if it is infinite. We show now that $\mathscr{E}^\frown$ is ι-realcompact. This will be achived by showing that there are no free weak maximal positive cones in $\mathbf{I}$ (see theorem 6).

Theorem 15. *The algebra of bounded Baire functions over $\mathscr{E}^\frown$ possesses no free weak maximal positive cones. Thus the space $\mathscr{E}^\frown$ is ι-realcompact.*

Proof. Suppose $\mathfrak{C}$ is a free w. m. p. c. in $\mathbf{I}$. Then it is easy to see that $\mathfrak{C} \cap \mathbf{B}^\frown$ is a m. p. c. in $\mathbf{B}^\frown$. (Proof. Since $\mathfrak{C}$ is a m. p. c., then $\varphi \in \mathbf{I}$ implies by the proof of theorem 2 that there exists a unique number λ such that $|\varphi - \lambda e^\frown|$ is in $\mathfrak{C}$. It is easy to see that $\mathfrak{C} \cap \mathbf{B}^\frown$ is a p. c. in $\mathbf{B}^\frown$. Suppose it is not maximal. Then there exists a m. p. c. $\mathfrak{D}$ in $\mathbf{B}^\frown$ properly including $\mathfrak{C} \cap \mathbf{B}^\frown$. Thus there is in $\mathfrak{D}$ a function $f^\frown \in \mathbf{B}^\frown$ with $f^\frown \notin \mathfrak{C} \cap \mathbf{B}^\frown$. Now for some $\lambda \neq 0$, $|f^\frown - \lambda e^\frown|$ is in $\mathfrak{C}$ hence in $\mathfrak{C} \cap \mathbf{B}^\frown$, hence in $\mathfrak{D}$. Since $|f^\frown - \lambda e^\frown|$ and $f^\frown$ are in the m. p. c. $\mathfrak{D}$, $|\lambda| e^\frown$ is in $\mathfrak{D}$ which is absurd. Hence $\mathfrak{C} \cap \mathbf{B}^\frown$ is a m. p. c.) Since $\mathscr{E}^\frown$ is β-compact, there exist a point $x_0 \in \mathscr{E}^\frown$ such that $\mathfrak{C} \cap \mathbf{B}^\frown$ consists precisely of the positive singular functions $f^\frown \in \mathbf{B}^\frown$ such $f^\frown(x_0) = 0$.

Since $\mathfrak{C}$ is free, there is a Baire function φ in $\mathfrak{C}$ such that $\varphi(x_0) > 0$. The set $\{x : \varphi(x) = \varphi(x_0)\}$ is a Baire set, hence contains a zero set $\mathfrak{Z}$ such that $x_0 \in \mathfrak{Z}$. Let $f^\frown \geqq 0$ be so chosen that $\mathfrak{Z}(f^\frown) = \mathfrak{Z}$. Then $f^\frown \in \mathfrak{C} \cap \mathbf{B}^\frown$. Thus $f^\frown + \varphi \in \mathfrak{C}$ and also $(f^\frown + \varphi) \rhd 0$ on $\mathscr{E}^\frown$ hence $\mathfrak{C}$ is not weak. This contradiction shows that $\mathfrak{C}$ does not exist. Hence $\mathscr{E}^\frown$ is τ-realcompact.

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A new proof of Plancherel's theorem for locally compact abelian groups

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§ 1. Introduction

Let G be a locally compact Abelian [LCA] group, with character group X. The famous theorem of PLANCHEREL—WEIL—KREĬN asserts that the Fourier transformation is a unitary mapping of $\mathfrak{L}_2(G)$ onto $\mathfrak{L}_2(X)$, the usual Fourier transformation being extended by continuity from the subspace $\mathfrak{L}_1(G) \cap \mathfrak{L}_2(G)$. The proofs given by KREĬN [see for example [2], § 31, N° 4] and WEIL [[4], pp. 113—118] use a number of delicate theorems of functional analysis. It seems worth while to give a completely elementary proof. Our argument is modelled on the beautiful proof given by F. RIESZ [3] for the classical case in which G is the additive group of real numbers. To apply RIESZ's idea, we need an analogue of the sequence of functions $\exp\left(\frac{-1}{2n^2}x^2\right)$ $(n=1, 2, \ldots)$ for an arbitrary LCA group containing a compact open subgroup. In § 2, we construct these functions quite explicitly. The proof of PLANCHEREL's theorem is then short, and is given in § 3. We also show that the "reverse" Fourier transformation is the inverse of the direct one.

Our notation is the following. For $f \in \mathfrak{L}_1(G)$ and $\psi \in X$, we write

$$Tf(\psi) = \int_G f(t)\overline{\psi(t)}\,dt.$$

That is, we write the Fourier transform of f as Tf. For $g \in \mathfrak{L}_1(X)$ and $s \in G$, we write

$$T^*g(s) = \int_X g(\psi)\psi(s)\,d\psi.$$

For functions $f_1, f_2 \in \mathfrak{L}_1(G)$, we write $f_1 * f_2$ for the convolution product of f_1 and f_2: $f_1 * f_2(s) = \int_G f_1(st^{-1})f_2(t)\,dt$. All integrals are with respect to Haar measure. For an integer $a > 1$, let R^a denote a-dimensional real Euclidean space. We write elements of R^a as $\mathbf{u} = (u_1, u_2, \ldots, u_a)$, etc. For a finite set B, let $\nu(B)$ denote the number of elements in the set B. The characteristic function of a set B is denoted by the symbol ξ_B [it will be clear from the context what the domain of ξ_B is].

We need the following facts about LCA groups, all of which are classical. [Complete proofs are found, for example, in [1].]

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(1. 1) A LCA group G is topologically isomorphic with a direct product $R^a \times H$ where R is the additive group of real numbers, a is a nonnegative integer, and H is a LCA group containing a compact open subgroup J.

(1. 2) Let H be as in (1. 1) and let Υ be the character group of H. The annihilator A of J in Υ is a compact open subgroup of Υ.

(1. 3) The Pontryagin-van Kampen duality theorem holds for H and Υ. That is, every continuous character of Υ has the form $\chi \to \chi(x)$ for some $x \in H$. Furthermore, J is the annihilator of A in H.

§ 2. Construction of auxiliary functions

We here make some preliminary constructions. The key to our whole argument is the following elementary fact.

(2. 1) Lemma. *Let S be a countable Abelian group. There is a sequence $\{P_n\}_{n=1}^{\infty}$ of finite subsets of S such that*

$$P_1 \subset P_2 \subset \cdots \subset P_n \subset \cdots, \quad P_n = P_n^{-1}, \quad \bigcup_{n=1}^{\infty} P_n = S, \quad \textit{and}$$

$$\text{(i)} \qquad \lim_{n\to\infty} \frac{\nu((xP_n) \cap P_n)}{\nu(P_n)} = 1$$

for all $x \in S$.[2])

Proof. (I) Suppose that A is any finite subset of S and that $x \in S$. Write $A_n = \bigcup_{k=0}^{n} x^k A$ $(n=0, 1, 2, \ldots)$ and $B_n = A_n \cap A'_{n-1}$ $(n=1, 2, 3, \ldots)$. Then it is evident that $B_n \subset xB_{n-1}$ $(n=2, 3, 4, \ldots)$, so that

$$\text{(1)} \qquad \nu(A) \geqq \nu(B_1) \geqq \nu(B_2) \geqq \cdots \geqq \nu(B_n) \geqq \cdots,$$

and

$$\text{(2)} \qquad \frac{\nu((xA_n) \cap A'_n)}{\nu(A_n)} = \frac{\nu(B_{n+1})}{\nu(A) + \nu(B_1) + \cdots + \nu(B_n)}$$

for $n = 1, 2, \ldots$. If $B_{n+1} \neq \emptyset$, (2) and (1) show that

$$\text{(3)} \qquad \frac{\nu((xA_n) \cap A_n)}{\nu(A_n)} \leqq \frac{1}{n+1},$$

and (3) follows trivially from (2) if $B_{n+1} = \emptyset$.

(II) Now let U and V be any finite subsets of S such that $e \in U \cap V$ [e is the identity of S], $V = V^{-1}$, and let ε be a positive real number. Then we can find a finite subset P of S such that $P = P^{-1}$, $V \subset P$, and

$$\text{(4)} \qquad \frac{\nu((UP) \cap P')}{\nu(P)} < \varepsilon.$$

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[2]) This lemma is closely related to although it is not a special case of Lemma (18.13) of [1].

To do this, write the set $U \cup U^{-1}$ as $\{t_1, t_2, \dots, t_r\}$ and for $n = 1, 2, 3, \dots$, let $W_n = \bigcup \{t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_r^{\alpha_r} V\}$, the union being taken over all ordered r-tuples $(\alpha_1, \alpha_2, \dots, \alpha_r)$ of nonnegative integers such that $\alpha_j \leqq n$ for $j = 1, 2, \dots, r$. It is plain that

$$(5) \qquad UW_n \subset \bigcup_{j=1}^{r} t_j W_n.$$

Now for a fixed $j \in \{1, 2, \dots, r\}$, we have $W_n = \bigcup_{k=0}^{n} t_j^k A$, where $A = \bigcup \{t_1^{\alpha_1} \cdots t_{j-1}^{\alpha_{j-1}} t_{j+1}^{\alpha_{j+1}} \cdots \cdots t_r^{\alpha_r} V\}$. From (3) we infer that

$$(6) \qquad \frac{\nu((t_j W_n) \cap W_n')}{\nu(W_n)} \leqq \frac{1}{n+1}$$

for $n = 1, 2, 3, \dots$. Using (5) and (6), we have

$$\frac{\nu((UW_n) \cap W_n')}{\nu(W_n)} \leqq \sum_{j=1}^{r} \frac{\nu((t_j W_n) \cap W_n')}{\nu(W_n)} \leqq \frac{r}{n+1}.$$

Now take P as any W_n with $n > \frac{r}{\varepsilon} - 1$.

(III) In completing the present proof, we may obviously suppose that S is infinite; and then we write $S = \{x_1, x_2, \dots, x_n, \dots\}$, where $x_1 = e$ and the x_n's are all distinct. For $n = 1, 2, 3, \dots$, let $U_n = \{x_1, x_2, \dots, x_n\} \cup \{x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}\}$. We define the sets P_n by induction. Let $P_1 = \{x_1\}$ and suppose that $P_2, P_3, \dots, P_{n-1}$ have been defined. Use part (II) to find a set P_n such that $P_n = P_n^{-1}$, $P_n \supset P_{n-1} \cup U_n$, and

$$\frac{\nu((U_n P_n) \cap P_n')}{\nu(P_n)} < \frac{1}{n}.$$

Each $x \in S$ is some x_m; and so for $n \geqq m$, we have

$$\frac{\nu((xP_n) \cap P_n')}{\nu(P_n)} \leqq \frac{\nu((U_n P_n) \cap P_n')}{\nu(P_n)} < \frac{1}{n},$$

and

$$\frac{\nu((xP_n) \cap P_n)}{\nu(P_n)} = 1 - \frac{\nu((xP_n) \cap P_n')}{\nu(P_n)} > 1 - \frac{1}{n}.$$

This establishes (i).

(2.2) Theorem. *Let H be a LCA group containing a compact open subgroup J. Let Y be the character group H and let A be the annihilator of J in Y. Let Haar measures on H and Y be chosen so that both J and A have measure 1. Let Δ be any σ-compact subset of Y. There is then a sequence $\{w_n\}_{n=1}^{\infty}$ of functions on Y with the following properties.*

(i) *Each w_n is continuous and vanishes outside of a compact set, and $w_n(Y) \subset [0, 1]$.*

(ii) *The sequence $\{w_n\}_{n=1}^{\infty}$ is increasing and $\lim_{n\to\infty} w_n \geqq \xi_\Delta$.*

(iii) *Write $\varphi_n(x) = \int_Y w_n(\chi)\chi(x)\,d\chi$ for $x \in H$. Then we have $\varphi_n \geqq 0$ and $\int_H \varphi_n(x)\,dx = 1$.*

(iv) *If* $\chi \in \Delta$, *then* $\lim_{n\to\infty} \int_H \varphi_n(x^{-1}y)\chi(y)dy = \chi(x)$ *for all* $x \in H$. *If* $\chi \in \mathrm{Y} \cap \Delta'$, *then* $\int_H \varphi_n(x^{-1}y)\chi(y)dy = 0$ *for all positive integers* n.

Proof. As noted in (1. 2), A is a compact open subgroup of Y. With no loss of generality we suppose that Δ is a subgroup of Y that is a countably infinite union of distinct cosets of A; we write $\Delta = \bigcup_{k=1}^{\infty} \chi_k \mathrm{A}$, where χ_1 is the character identically equal to 1 and $\chi_k\chi_l^{-1} \notin \mathrm{A}$ if $k \neq l$. We apply (2. 1) to the countable group Δ/A. Let τ be the natural mapping of Δ onto Δ/A; let $\{P_n\}_{n=1}^{\infty}$ be the sets constructed for the group Δ/A in (2. 1); and let $\mathrm{B}_n = \tau^{-1}(P_n)$ $(n = 1, 2, 3, \ldots)$. Clearly each set B_n is a finite union of cosets of A and is thus a compact open set; suppose that B_n is the union of r_n distinct cosets of A. Upon renumbering the characters $\{\chi_k\}_{k=1}^{\infty}$ if necessary, we may suppose that there is a sequence $1 = r_1 < r_2 < \cdots < r_n < \cdots$ of positive integers such that $\mathrm{B}_n = \bigcup_{k=1}^{r_n} \chi_k \mathrm{A}$ $(n = 1, 2, 3, \ldots)$.

We define w_n by

$$w_n = \frac{1}{r_n} \xi_{\mathrm{B}_n} * \xi_{\mathrm{B}_n} \qquad (n = 1, 2, 3, \ldots).$$

It is clear that $w_n(\chi)$ is equal to $\frac{1}{r_n}$ times the Haar measure of the set $(\chi^{-1}\mathrm{B}_n) \cap \mathrm{B}_n$. For $\chi \notin \Delta$, we have $(\chi^{-1}\mathrm{B}_n) \cap \mathrm{B}_n = \varnothing$. For $\alpha \in \mathrm{A}$ and $m = 1, 2, 3, \ldots$, it is easy to see that

$$w_n(\chi_m\alpha) = \frac{\nu\big((\tau(\chi_m^{-1})P_n) \cap P_n\big)}{\nu(P_n)}.$$

This equality and (2. 1. i) imply that $\lim_{n\to\infty} w_n(\chi_m\alpha) = 1$. We have thus proved (i) and (ii).

Since J is the annihilator of A (1. 3), an easy computation shows that

$$\int_{\mathrm{B}_n} \chi(x)\, d\chi = \sum_{k=1}^{r_n} \chi_k(x)\xi_J(x) \qquad (x \in H).$$

Since $\mathrm{B}_n = \mathrm{B}_n^{-1}$ [this follows from the equality $P_n = P_n^{-1}$ and the fact that A is a subgroup of Y], we have

$$\varphi_n(x) = \frac{1}{r_n}\left| \int_{\mathrm{B}_n} \chi(x)\, d\chi \right|^2 = \frac{1}{r_n}\left| \sum_{k=1}^{r_n} \chi_k(x) \right|^2 \xi_J(x).$$

Thus φ_n is nonnegative. Since $\chi_k\chi_l^{-1} \in \mathrm{A}$ only for $k = l$, we also have

$$\int_H \varphi_n(x)\, dx = \frac{1}{r_n} \sum_{k=1}^{r_n} \sum_{l=1}^{r_n} \int_J \chi_k(x)\chi_l^{-1}(x)\, dx = 1.$$

Thus we have proved (iii).

Now let χ be any character in Δ. Then $\chi=\chi_m\alpha$ for a unique positive integer m and a unique $\alpha\in\Delta$. We have

$$\int_H \varphi_n(x^{-1}y)\chi(y)\,dy = \int_H \varphi_n(y)\chi(xy)\,dy = \chi(x)\int_H \varphi_n(y)\chi(y)\,dy =$$

$$= \chi(x)\frac{1}{r_n}\sum_{k=1}^{r_n}\sum_{l=1}^{r_n}\int_J \chi_k(x)\chi_l^{-1}(x)\chi_m(x)\,dx.$$

The integral $\int_J \chi_k(x)\chi_l^{-1}(x)\chi_m(x)\,dx$ is 0 if $\chi_k\chi_l^{-1}\chi_m\notin A$ and is 1 otherwise. The number of pairs (k,l) for which $\chi_k\chi_l^{-1}\chi_m\in A$ is equal to $\nu((\tau(\chi_m)P_n)\cap P_n)$. Thus (2. 1) implies that $\lim_{n\to\infty}\int_H \varphi_n(x^{-1}y)\chi(y)\,dy=\chi(x)$. If $\chi\notin\Delta$, then $\chi_k\chi_l^{-1}\chi$ is in A for no choice of k and l; hence $\int_H \varphi_n(x^{-1}y)\chi(y)\,dy=0$ in this case. This establishes (iv) and completes the present proof.

(2. 3) Theorem. *Let H, J, Y, and A be as in (2. 2). Let g be a continuous function on H vanishing outside of a compact set F. Let Γ be a σ-compact subset of Y. Then there is an open σ-compact subgroup Δ of Y such that $\Delta\supset\Gamma$ and such that the functions φ_n constructed for Δ as in* (2. 2. iii) *have the property that*

(i) $$\lim_{n\to\infty}\int_H \varphi_n(x^{-1}y)g(y)\,dy = g(x)\qquad (x\in H).$$

Proof. Consider the open compact subset JF of H. The Stone—Weierstrass theorem implies that there is a countable subset Λ of Y such that complex linear combinations of characters in Λ approximate g arbitrarily in the uniform metric on JF. Let Δ be any σ-compact subgroup of Y that contains $A\cup\Gamma\cup\Lambda$. Let ε be a positive real number and let

(1) $$\left|g(z)-\sum_{j=1}^{m}a_j\chi_j(z)\right|<\frac{\varepsilon}{3}$$

for all $z\in JF$, where the a_j are complex numbers and $\chi_j\in\Lambda$. We have $\int\varphi_n(x^{-1}y)g(y)\,dy=0$ for $x\notin JF$, so that (i) holds trivially for all $x\notin JF$. For n sufficiently large, (2. 2. iv) implies that

(2) $$\left|\int_H \varphi_n(x^{-1}y)\left(\sum_{j=1}^{m}a_j\chi_j(y)\right)dy-\sum_{j=1}^{m}a_j\chi_j(x)\right|<\frac{\varepsilon}{3}$$

for all $x \in JF$. Finally we have

$$(3) \qquad \left| \int_H \varphi_n(x^{-1}y) g(y)\, dy - \int_H \varphi_n(x^{-1}y) \left(\sum_{j=1}^{m} a_j \chi_j(y) \right) dy \right| =$$

$$= \left| \int_J \varphi_n(y) \left[g(xy) - \sum_{j=1}^{m} a_j \chi_j(xy) \right] dy \right| \leqq$$

$$\leqq \int_J \varphi_n(y)\, dy \cdot \max \left\{ \left| g(xy) - \sum_{j=1}^{m} a_j \chi_j(xy) \right| : x \in JF,\ y \in J \right\} < \frac{\varepsilon}{3}.$$

Combining (1), (2), and (3), we obtain (i).

§ 3. Proof of Plancherel's theorem

Throughout this section, G is an arbitrary LCA group and X is its character group.

(3.1) Theorem. *Let f be a function in $\mathfrak{L}_1(G) \cap \mathfrak{L}_2(G)$. Then Tf is in $\mathfrak{L}_2(X)$ and for an appropriate choice of Haar measure on G and X, we have*

$$\text{(i)} \qquad \|Tf\|_2 \leqq \|f\|_2 .$$

Proof. Let G be represented as $R^a \times H$ as in (1.1), so that X is represented as $R^a \times Y$, where Y is the character group of H. A generic [continuous!] character of $R^a \times H$ has the form

$$(\mathbf{u}, x) \to \exp\left[i(u_1 v_1 + \cdots + u_a v_a)\right] \chi(x)$$

for some $\mathbf{u} \in R^a$ and $\chi \in Y$. Let J be a compact open subgroup of H and A the annihilator of J in Y. We choose Haar measure in G to be the product of $(2\pi)^{-a/2}$ times Lebesgue measure on R^a and of the Haar measure on H assigning measure 1 to J. We choose Haar measure on X to be the product of $(2\pi)^{-a/2}$ times Lebesgue measure on R^a and of the Haar measure on Y assigning measure 1 to A.

The function Tf is continuous on X and vanishes at infinity, so that Tf vanishes outside of a set $R^a \times \Delta$, where Δ is a σ-compact subset of Y as in (2.2). Let $\{w_n\}_{n=1}^{\infty}$ be a sequence of functions on Y as constructed in (2.2) for this set Δ. Define the sequence $\{W_n\}_{n=1}^{\infty}$ of functions on $X = R^a \times Y$ by

$$(1) \qquad W_n(\mathbf{v}, \chi) = \exp\left[-\frac{1}{2n^2} (v_1^2 + \cdots + v_a^2) \right] w_n(\chi)$$

[with obvious modifications if the factor R^a or the factor Y is missing]. Defining Φ_n on $G = R^a \times H$ by

$$(2) \qquad \Phi_n(\mathbf{u}, x) = \int_{R^a \times Y} W_n(\mathbf{v}, \chi) \exp\left[i(u_1 v_1 + \cdots + u_a v_a)\right] \chi(x)\, d(\mathbf{v}, x),$$

we have

$$\text{(3)} \qquad \Phi_n(\mathbf{u}, x) = (2\pi)^{-a/2} \prod_{k=1}^{a} \int_{-\infty}^{\infty} \exp\left[\frac{-1}{2n^2} v_k^2 + iu_k v_k\right] dv_k \int_{Y} \chi(x) w_n(\chi)\, d\chi$$

$$= n^a \left(\prod_{k=1}^{a} \exp\left[-\frac{n^2}{2} u_k^2\right]\right) \varphi_n(x).$$

It is obvious from (3) that

$$\int_{R^a \times H} \Phi_n(\mathbf{u}, x)\, d(\mathbf{u}, x) = n^a \left(\prod_{k=1}^{a} (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp\left[\frac{-n^2}{2} u_k^2\right] du_k \int_{H} \varphi_n(x)\, dx \right) = 1.$$

Also each function Φ_n is plainly nonnegative.

For notational convenience, we now revert to one-variable notation in writing integrals over G and X. By Fubini's theorem [which evidently applies], we have

$$\text{(4)} \qquad \int_X |Tf(\psi)|^2 W_n(\psi)\, d\psi = \int_X \int_G f(t)\overline{\psi(t)}\, dt \int_G \overline{f(s)}\, \psi(s)\, ds W_n(\psi)\, d\psi$$

$$= \int_G \int_G \int_X W_n(\psi)\psi(t^{-1}s)\, d\psi f(t) \overline{f(s)}\, dt\, ds$$

$$= \int_G \int_G \Phi_n(t^{-1}s) f(t) \overline{f(s)}\, dt\, ds.$$

Applying the Cauchy–Schwarz inequality to the last integral, and taking cognizance of the invariance of the Haar integral, we have

$$\text{(5)} \qquad \int_G \int_G \Phi_n(t^{-1}s) f(t) \overline{f(s)}\, dt\, ds \leqq$$

$$\leqq \left[\int_G \int_G \Phi_n(t^{-1}s) |f(t)|^2\, ds\, dt\right]^{\frac{1}{2}} \times \left[\int_G \int_G \Phi_n(t^{-1}s) |f(s)|^2\, ds\, dt\right]^{\frac{1}{2}} = \|f\|_2^2.$$

Combining (4) and (5) and taking the limit as $n \to \infty$, we obtain (i).

Theorem (3. 1) shows that the Fourier transformation T, which is linear on $\mathfrak{L}_1(G) \cap \mathfrak{L}_2(G)$, carries this space into $\mathfrak{L}_2(X)$ without increasing the $\mathfrak{L}_2$ norm. Therefore there is a unique, linear, norm nonincreasing mapping of $\mathfrak{L}_2(G)$ into $\mathfrak{L}_2(X)$ that extends T. We call this extended mapping T as well, and we note that if $\|f_n - f\|_2 \to 0$, where $f_n \in \mathfrak{L}_1(G) \cap \mathfrak{L}_2(G)$ and $f \in \mathfrak{L}_2(G)$, then $\|Tf_n - Tf\|_2 \to 0$.

(3. 2) Theorem. *Let g be a function in $\mathfrak{L}_1(X) \cap \mathfrak{L}_2(X)$. Then T^*g is in $\mathfrak{L}_2(G)$, and if Haar measures on G and X are chosen as in* (3. 1), *we have*

(i) $$\|T^*g\|_2 \leqq \|g\|_2.$$

Proof. This assertion is proved just as (3. 1) was proved. Plainly the integral $\int_{X} g(\psi)\psi(x)\,d\psi$ behaves just like the integral $\int_{G} f(x)\overline{\psi(x)}\,dx$. Since the annihilator of A in G is J (1. 3) and since Haar measures in G and X have been chosen symmetrically, the proof of (3. 1) can be repeated *verbatim* to yield the present theorem.

Like T, the transformation T^* can be extended to a linear norm nonincreasing mapping of $\mathfrak{L}_2(\mathrm{X})$ into $\mathfrak{L}_2(G)$.

(3. 3) Theorem. *For* $f \in \mathfrak{L}_2(G)$, *we have*

(i) $$T^* Tf = f.$$

Proof. We lose no generality in proving the theorem for $f \in \mathfrak{L}_1(G) \cap \mathfrak{L}_2(G)$. At first we write G in the form $R^a \times H$ as in (1. 1), and we consider any function h on $R^a \times H$ of the form

$$h(\mathbf{u}, x) = g_1(u_1)g_2(u_2)\cdots g_a(u_a)g(x),$$

where $g_1, g_2, \ldots, g_a, g$ are continuous on $R, R, \ldots, R, H$ respectively, and vanish outside of compact sets. It is elementary and easy to show that

$$\lim_{n\to\infty} \left\{(2\pi)^{-1/2} n \int_{-\infty}^{\infty} \exp\left[-\frac{n^2}{2}(u-v)^2 g(v)\right] dv\right\} = g(u)$$

for all $u \in R$ if g is bounded and uniformly continuous on R. We now construct a subset Δ of Y containing the set $\{\chi \in \mathrm{Y} : Tf(\mathbf{v}, \chi) \neq 0$ for some $\mathbf{v} \in R^a\}$ and having also the property that (2. 3. i) holds for the function g. Then we form the functions W_n and Φ_n as in (3. 1. 1) and (3. 1. 2) *for this choice of* Δ. We see immediately that

(1) $$\lim_{n\to\infty} \int_{R^a \times H} \Phi_n(-\mathbf{u}+\mathbf{v}, x^{-1}y)h(\mathbf{v}, y)\,d(\mathbf{v}, y) = h(\mathbf{u}, x)$$

for all $(\mathbf{u}, x) \in R^a \times H$. Reverting to one-variable notation for G and X, we write

(2) $$\int_G \int_X W_n(\psi)Tf(\psi)\psi(s)\,d\psi h(s)\,ds = \int_G \int_X W_n(\psi) \int_G f(t)\psi(t^{-1})\,dt\psi(s)\,d\psi h(s)\,ds =$$
$$= \int_G f(t) \int_G \int_X W_n(\psi)\psi(t^{-1}s)\,d\psi h(s)\,ds\,dt = \int_G f(t) \int_G \Phi_n(t^{-1}s)h(s)\,ds\,dt.$$

By (1), the integral $\int_G \Phi_n(t^{-1}s)h(s)ds$ converges [boundedly!] to $h(t)$ for all $t \in G$. The integral $\int_X W_n(\psi)Tf(\psi)\psi(s)d\psi$ is equal to $T^*(W_n Tf)$. Since T^* is linear and norm nonincreasing (3. 2), and $\|W_n Tf - Tf\|_2 \to 0$, we have $\lim\limits_{n\to\infty} T^*(W_n Tf) = T^*(Tf)$

in the $\mathfrak{L}_2(G)$ metric. The equalities (2) then imply that

$$\int_G T^*Tf(s)h(s)\,ds = \int_G \lim_{n\to\infty} T^*(W_nTf)(s)h(s)\,ds = \tag{3}$$

$$= \lim_{n\to\infty}\int_G T^*(W_nTf)(s)h(s)\,ds = \int_G f(t)h(t)\,dt.$$

Linear combinations of functions h are dense in $\mathfrak{L}_2(G)$, and so (3) implies that $T^*Tf=f$. Since $\mathfrak{L}_1(G)\cap\mathfrak{L}_2(G)$ is dense in $\mathfrak{L}_2(G)$, this equality holds for all $f\in\mathfrak{L}_2(G)$.

(3. 4) Theorem. *The Fourier transformation T is a linear isometry of $\mathfrak{L}_2(G)$ onto $\mathfrak{L}_2(X)$, and T^*T and TT^* are the identity transformations on $\mathfrak{L}_2(G)$ and $\mathfrak{L}_2(X)$ respectively.*

Proof. For $g\in\mathfrak{L}_2(X)$, we have $T^*g\in\mathfrak{L}_2(G)$ by (3. 2). We apply (3. 3) [with the rôles of G and X interchanged] to infer that $TT^*g=g$. Hence T carries $\mathfrak{L}_2(G)$ onto $\mathfrak{L}_2(X)$. For $f\in\mathfrak{L}_2(G)$, we have

$$\|f\|_2=\|T^*Tf\|_2\leqq\|Tf\|_2\leqq\|f\|_2.$$

Thus T preserves norms and so is an isometry.

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Über die Konvergenz von Partialsummen der Orthogonalreihen

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1. Es sei $\{\varphi_n(x)\}$ ein im Grundintervall $[a, b]$ orthonormiertes Funktionensystem und $\{a_n\}$ eine Koeffizientenfolge mit $\sum a^2 < \infty$. A. N. KOLMOGOROFF hat den folgenden Satz bewiesen [1]:

Ist die Orthogonalreihe

$$\sum_{n=0}^{\infty} a_n \varphi_n(x) \tag{1}$$

fast überall $(C, 1)$*-summierbar, so ist die Folge der Partialsummen*

$$s_{2^n}(x) = \sum_{\nu=0}^{2^n} a_\nu \varphi_\nu(x) \tag{2}$$

fast überall konvergent.

Wenn man voraussetzt, daß die Koeffizientenfolge $\{a_n\}$ „im wesentlichen monoton abnehmend" ist, dann kann man die entsprechende Aussage über einen allgemeineren Typus von Teilfolgen behaupten. Diese Teilfolgen sind — im Gegensatz zu (2) — nicht mehr stark lückenhaft.[1]) Es gilt nämlich der folgende Satz von K. TANDORI [2]:

Es sei $N(\geqq 1)$ *eine beliebig angegebene natürliche Zahl. Mit* $\{n_k\}$ *wird die (wachsend angeordnete) Folge derjenigen natürlichen Zahlen bezeichnet, die in der Form*

$$n_k = 2^{\nu_1} \pm 2^{\nu_2} \pm \dots \pm 2^{\nu_r} \tag{3}$$

mit ganzzahligen Exponenten $\nu_1 > \nu_2 > \dots > \nu_r \geqq 0$ $(1 \leqq r \leqq N)$ *aufgeschrieben werden können. Sind die Bedingungen*

$$c_n \geqq c_{n+1} (>0), \quad \Sigma c_n^2 < \infty, \quad a_n = O(c_n) \tag{4}$$

erfüllt, und ist die Reihe (1) *fast überall* $(C, 1)$*-summierbar, so konvergiert die Folge der* n_k*-ten Partialsummen der Reihe* (1) *fast überall.*

2. In dieser Note geben wir eine einfache Bedingung dafür, daß die im Tandorischen Satz auftretende Folge $\{s_{n_k}\}$ auch in dem Fall fast überall konvergent sei, wenn wir über die Monotonität der Koeffizientenfolge $\{a_n\}$ gar nichts voraussetzen.

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[1]) Wir nennen eine Indexfolge $\{\nu_n\}$ stark lückenhaft, wenn für jedes n $\dfrac{\nu_{n+1}}{\nu_n} \geqq q > 1$ gilt.

Satz. *Ist*

$$\sum_{n=3}^{\infty} a_n^2 (\log\log n)^2 < \infty, \tag{5}$$

so konvergiert die Teilfolge $\{s_{n_k}(x)\}$ *der Partialsummen der Orthogonalreihe* $\sum a_n\varphi_n(x)$ *fast überall.*

Beweis. Auf Grund der Bedingung (5) ist die Orthogonalreihe (1) fast überall $(C, 1)$-summierbar [3], [4]. Nach dem genannten Kolmogoroffschen Satz konvergiert also die Teilfolge der Partialsummen $\{s_{2^\nu}(x)\}$ fast überall.

Betrachten wir nun die wachsend angeordnete Folge der Zahlen (3), und es sei $\{k_\nu\}$ diejenige Teilfolge der Indices $\{k\}$ für welche

$$n_{k_\nu} = 2^\nu.$$

Unter Benützung dieser Schreibweise können wir also sagen, daß die Folge

$$s_{n_{k_\nu}}(x) = \sum_{\varrho=0}^{n_{k_\nu}} a_\varrho \varphi_\varrho(x)$$

fast überall konvergent ist, und daher haben wir nur die folgende Aussage zu prüfen:

Wenn i ein beliebiger Index zwischen k_ν und $k_{\nu+1}$ ist, dann besteht für $\nu \to \infty$ die Relation

$$s_{n_i}(x) - s_{n_{k_\nu}}(x) = o_x(1)$$

fast überall.

Zu diesem Zweck bilden wir für jedes ν die folgende Summe:

$$\begin{aligned}\sum_{i=k_\nu+1}^{k_{\nu+1}} \left(s_{n_i}(x) - s_{n_{i-1}}(x)\right) &= \sum_{i=k_\nu+1}^{k_{\nu+1}} \sum_{l=n_{i-1}+1}^{n_i} a_l \varphi_l(x) = \\ = \sum_{i=k_\nu+1}^{k_{\nu+1}} \sqrt{\sum_{l=n_{i-1}+1}^{\nu_l} a_l^2} &\left\{ \frac{\sum_{l=n_{i-1}+1}^{n_i} a_l \varphi_l(x)}{\sqrt{\sum_{l=n_{i-1}+1}^{n_i} a_l^2}} \right\} \equiv \sum_{i=k_\nu+1}^{k_{\nu+1}} C_i \Phi_i(x),\end{aligned} \tag{6}$$

wobei $\{\Phi_i(x)\}_{i=k_\nu+1}^{k_{\nu+1}}$ ein Orthonormalsystem in $[a, b]$ ist[3]). Nach dem bekannten Menchoffschen Lemma [5] existiert also für jedes ν eine nichtnegative Funktion $\delta_\nu(x)$, durch welche alle Abschnitte der Summe (6) dem Betrage nach majorisiert werden, d. h. für welche

$$|s_{n_i}(x) - s_{n_{k_\nu}}(x)| \leqq \delta_\nu(x) \tag{7}$$

[3]) Wir können ohne Beschränkung der Allgemeinheit annehmen, daß für jedes i die Summe $\sum_{l=n_{i-1}+1}^{n_i} a_l^2$ positiv ist.

für alle i zwischen k_ν und $k_{\nu+1}$ ist, und gleichzeitig gilt:

$$\int_a^b \delta_\nu^2(x)\,d\mu(x) = O\big(\log^2(k_{\nu+1}-k_\nu)\big)\sum_{i=k_\nu+1}^{k_{\nu+1}} C_i^2 = \tag{8}$$

$$= O\big(\log^2(k_{\nu+1}-k_\nu)\big)\sum_{i=k_\nu+1}^{k_{\nu+1}}\sum_{l=n_{i-1}+1}^{n_i} a_l^2 = O\big(\log^2(k_{\nu+1}-k_\nu)\big)\sum_{\varrho=n_{k_\nu}+1}^{n_{k_{\nu+1}}} a_\varrho^2 =$$

$$= O\big(\log^2(k_{\nu+1}-k_\nu)\big)\sum_{\varrho=2^\nu+1}^{2^{\nu+1}} a_\varrho^2.$$

Beachten wir aber, daß die Differenz $(k_{\nu+1}-k_\nu)$ — also die Anzahl derjenigen Elemente der Folge (3), die zwischen 2^ν und $2^{\nu+1}$ liegen — höchstens gleich der Potenz ν^N ist, so bekommen wir aus (8) die folgende Abschätzung:

$$\int_a^b \delta_\nu^2(x)\,d\mu(x) = O(1)(\log \nu^N)^2 \sum_{\varrho=2^\nu+1}^{2^{\nu+1}} a_\varrho^2 =$$

$$= O(1)(N\cdot\log\nu)^2\sum_{\varrho=2^\nu+1}^{2^{\nu+1}} a_\varrho^2 = O(1)(\log\log 2^\nu)^2\sum_{\varrho=2^\nu+1}^{2^{\nu+1}} a_\varrho^2.$$

Aus dieser Relation ergibt sich unmittelbar, daß wegen unserer Bedingung (5)

$$\sum_{\nu=2}^\infty \int_a^b \delta_\nu^2(x)\,d\mu(x) = O(1)\sum_{\nu=2}^\infty(\log\log 2^\nu)^2\sum_{\varrho=2^\nu+1}^{2^{\nu+1}} a_\varrho^2 =$$

$$= O(1)\sum_{n=4}^\infty a_n^2(\log\log n)^2 < \infty$$

gilt, und daher nach dem Beppo Levischen Satz $\sum \delta_\nu^2(x)$ fast überall konvergiert; also fast überall

$$\delta_\nu(x) = o_x(1) \qquad (\nu\to\infty)$$

gilt. Diese Relation liefert aber zusammen mit (7) die Richtigkeit unserer Behauptung.

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On some combinatorial relations concerning the symmetric random walk

By E. CSÁKI and I. VINCZE in Budapest

Dedicated to the three inseparable friends P. Erdős, T. Gallai, and P. Turán at the occasion of all being close to 50

§ 1. Introduction and notations

1. In this paper we shall consider sequences $\vartheta=(\vartheta_1, \vartheta_2, \ldots, \vartheta_{2n})$ of n $(+1)$-s and n (-1)-s, each possible array being of equal probability $1\Big/\binom{2n}{n}$. Thus the partial sums $s_0=0, s_i=\vartheta_1+\vartheta_2+\ldots+\vartheta_i \, (i=1,2,\ldots,2n)$ generate a simple symmetric random walk, returning after $2n$ steps to the origin.

We use the following notations:

$\varkappa=\max\limits_{0\leq i\leq 2n} s_i$; $\qquad \varrho=\min\{i : s_i=\varkappa\}$ (index of the first maximum).

$\lambda-1$ is the number of the intersections, i. e. the number of i-s with $s_i=0$, $s_{i-1}s_{i+1}=-1$ (thus λ is the number of half-waves).

γ is the Galton-statistics (i. e. 2γ is the number of indices i for which either $s_i>0$, or $s_i=0, s_{i-1}=+1$).

The authors have found the following asymptotic relation [6], [2]:

$$\lim_{n\to\infty} P(\varkappa<y\sqrt{2n},\ \varrho<2nz)=\lim_{n\to\infty} P(\lambda<y\sqrt{2n},\ \gamma<nz)=$$

$$=\sqrt{\frac{2}{\pi}}\int_0^y\int_0^z \frac{u^2}{[v(1-v)]^{3/2}}\, e^{-\frac{u^2}{2v(1-v)}}\,du\,dv \qquad (y\geq 0;\ 1\geq z\geq 0).$$

In connection with this relation, E. SPARRE ANDERSEN raised the question[1]), whether there exists some equivalence principle for the finite case too.

In the following we give some equivalence theorems and prove among others that

$$P(\varkappa=l)=\tfrac{1}{2}\big(P(\lambda=l)+P(\lambda=l+1)\big) \qquad (l=0,1,\ldots,n), \tag{1.1}$$

$$P(\varkappa=l,\ \varrho=r)=P(\lambda'=l,\ \pi=r), \tag{1.2}$$

[1]) At the occasion of the *Conference on Probability and Statistics* held in Oberwolfach, August 20—26, 1961.

where π denotes the number of positve terms in $(s_0, s_1, \ldots, s_{2n})$, while λ' is the number of indices i for which $s_{i-1}=0, s_i=+1$. (1.2) implies

$$P(\varrho=r)=P(\pi=r),$$

which is a special case of a well-known result of SPARRE ANDERSEN [1]; it implies also the following result of MIHALEVIČ [5]:

$$P(\varkappa=l) = P(\lambda'=l).$$

Thus *we have a joint equivalence between* $(\varkappa, \varrho)$ *and* (λ', π). We would like to point out furthermore that *each of our theorems is proved by means of one-to-one correspondences between the sets of paths considered.* This indicates a combinatorial and geometrical background of these equivalences.

We also remark that in our constructions $\varkappa$ appears virtually more as the number of ladder indices (see FELLER [3]), than as the maximum, both coinciding for the special variables $\vartheta_i = \pm 1$.

2. We shall make use of the following further terminology and notations:

The polygonal line whose subsequent vertices have the coordinat es (i, s_i) $(i=0, 1, 2, \ldots, j)$ is called the path $(s_0, s_1, \ldots, s_j)$.

E_{2n} is a path $(s_0, s_1, \ldots, s_{2n})$ with $s_{2n}=0$. A point $(2i, s_{2i})$ of the path E_{2n}, for which $s_{2i}=0$ and $s_{2i-1}s_{2i+1}=-1$, as well as the points $(0, 0)$ and $(2n, 0)$ of E_{2n}, are called intersection points or T-points. As defined above, $\lambda+1$ is the number of T-points.

By a T'-point we mean a point $(2i+1, 1)$ of the path E_{2n}, for which $s_{2i}=0$, $s_{2i+1}=+1$ (this kind of points was treated by MIHALEVIČ [5]). λ' is the number of T'-points.

E_{2n}^l is a path E_{2n} with $\lambda=l$,

$\langle i, j\rangle$ is a section of a path lying between the points (i, s_i) and (j, s_j), i. e. the sequence $(\vartheta_{i+1}, \vartheta_{i+2}, \ldots, \vartheta_j)$.

k is called a strict ladder index (FELLER [3]); if $s_k > s_i$ for $i=0, 1, \ldots, k-1$; k is called a strict backward ladder index if $s_k > s_i$ for $i=k, k+1, \ldots, 2n$.

A_r^l is a path $(s_0, s_1, \ldots, s_r)$, for which $s_0=0$, $s_1<l$, $s_2<l$, $\ldots$, $s_{r-1}<l$, $s_r=l$, i. e. its l-th strict ladder index being r.

$N(\cdot)$ is the number of all possible paths whose type is given in the brackets $\left(\text{e. g. } N(E_{2n})=\binom{2n}{n}\right)$.

§ 2. Equivalence relations

1. The maxium and the number of waves. We shall prove the following

Theorem 2.1. $P(\varkappa=l) = \frac{1}{2}\left[P(\lambda=l)+P(\lambda=l+1)\right]$ $(l=0, 1, 2, \ldots, n)$.

Proof. We consider a path E_{2n} with $\varkappa=l$. According to the index ϱ of the first maximum, we distinguish two different cases:

a) ϱ is the only position, for which the maximum takes place;

b) there are more than one maximum places.

In both cases we shall make use of the following

Lemma 2.1. $\frac{1}{2} N(E_{2n}^{l}) = N(A_{2n}^{2l})$.

This was proved in [2] by means of a one-to-one correspondence between the sets of paths.

In case a) we consider the sections $\langle 0, \varrho \rangle$ and $\langle \varrho, 2n \rangle$. Replacing in the second part the steps $(\vartheta_{\varrho+1}, \vartheta_{\varrho+2}, \ldots, \vartheta_{2n})$ by the steps $(-\vartheta_{2n}, -\vartheta_{2n-1}, \ldots, -\vartheta_{\varrho+2}, -\vartheta_{\varrho+1})$, we obtain a path A_{2n}^{2l}. According to Lemma 2. 1 this path can be transformed into a path E_{2n}^{l} with $s_1 = +1$.

Obviously this procedure is invertible, by considering the l-th strict ladder index of the path A_{2n}^{2l}.

In case b) let us denote by $\bar{\varrho}$ the index of the last maximum. The path E_{2n} with $s_{\varrho} = s_{\bar{\varrho}} = l$ consists of the following three sections: $\langle 0, \varrho \rangle$, $\langle \varrho, \bar{\varrho} \rangle$, $\langle \bar{\varrho}, 2n \rangle$. We apply the following transformation: we replace in $\langle \varrho, \bar{\varrho} \rangle$ the steps $(\vartheta_{\varrho+1} = -1, \vartheta_{\varrho+2}, \ldots, \vartheta_{\bar{\varrho}})$ by $(\vartheta_{\varrho+2}, \ldots, \vartheta_{\bar{\varrho}}, +1)$ and in $\langle \bar{\varrho}, 2n \rangle$ the steps $(\vartheta_{\bar{\varrho}+1}, \vartheta_{\varrho+2}, \ldots, \vartheta_{2n})$ by the steps $(-\vartheta_{2n}, -\vartheta_{2n-1}, \ldots, -\vartheta_{\bar{\varrho}+2}, -\vartheta_{\bar{\varrho}+1})$. Thus we obtain a path A_{2n}^{2l+2}. According to Lemma 2. 1 this path can be transformed into a path E_{2n}^{l+1} with $s_1 = +1$.

In order to invert this procedure we have only to find the l-th and $l+2$-th ladder indices of the path A_{2n}^{2l+1}. Cases a) and b) complete the proof of Theorem 2. 1.

2. Two variate equivalences. We shall prove the following

Theorem 2. 2. $P(\varkappa = l, \varrho = r) = P(\lambda' = l, \pi = r)$ $(l = 0, r = 0; l = 1, 2, \ldots, n, r = l, l+2, \ldots, 2n-l)$.

Proof[2]). For $r = 0$, $l = 0$ the paths of both kinds coincide, we have to consider only the case $l \geq 1$. Then each path with $(\lambda' = l, \pi = r)$ can be divided by the T'-points $(2i+1, 1)$ and the points $(2j, 0)$ with $s_{2j} = 0$ and $s_{2j-1} = +1$ into $2l$ or $2l+1$ sections, some of which are starting from $+1$ and ending in 0, all inner points being strictly positive (type α)), while the others are starting from 0, ending in $+1$, all inner points being non-positive (type β)).

The first section is always of type β); the last section is either of type α) or of type β), but in the latter case the last $(\vartheta_{2n+1} = +1)$ step is missing.

There are altogether l sections of type α) with total length r and l or $(l+1)$ sections of type β).

Let us now consider the sections of type α). We change all ϑ_i-s occurring in them into $-\vartheta_i$ and link together the new sections obtained by this procedure, maintaining their original order of succession. We now link together all sections of type β); denoting the steps of the section thus obtained by $(\vartheta'_{r+1}, \vartheta'_{r+2}, \ldots, \vartheta'_{2n})$ we transform them into $(-\vartheta'_{2n}, -\vartheta'_{2n-1}, \ldots, -\vartheta'_{r+2}, -\vartheta'_{r+1})$ and join the respective section to the first section obtained. As a result we obtain a path with $\varkappa = l$, $\varrho = r$.

The reverse procedure transforms each path $\{\varkappa = l, \varrho = r\}$ into the corresponding path $\{\lambda' = l, \pi = r\}$; this can be performed by considering the strict ladder indices in section $\langle 0, r \rangle$ and the strict backward ladder indices in section $\langle r, 2n \rangle$.

[2]) Similar construction is used by CH. HOBBY and R. PYKE [4].

In the following theorem we shall prove two equivalences according to whether the maximum is even or odd.

Theorem 2. 3.

$$P(\varkappa=s_{2r'}=2l)=\tfrac{1}{2}P(\lambda=2l,\gamma=r')+P(\lambda=2l+1,\gamma=r')+\tfrac{1}{2}P(\lambda=2l+2,\gamma=r')$$

and $$P(\varkappa=s_{2r'+1}=2l-1)=$$

$$=P(\lambda=2l-1,\gamma=r';s_1=-1)+P(\lambda=2l,\gamma=r')+P(\lambda=2l+1,\gamma=r',s_1=+1).$$

Proof. We use the same procedure as in the proof of Lemma 2. 1.

The crucial point in the proof of Lemma 2. 1 was the division of a path A_{2n}^{2l} by means of its even strict ladder indices. The last step of each section between two consecutive ladder indices is always $(+1)$; omitting this and placing a (-1) before the section, we obtain a negative half wave.

Considering a path whose maximum $2l$ is taken on for the index $2r'$ let us denote by $2r$ $(2\bar{r})$ the first (last) index of maximum. The section $\langle 0, 2r\rangle$ is a path A_{2r}^{2l}, the section $\langle 2\bar{r}, 2n\rangle$ is an inverted path $A_{2(n-\bar{r})}^{2l}$. As described before, both sections can be divided into l parts and each part can be transformed into a negative half wave. The half waves generated by A_{2r}^{2l} will be turned into positive half waves by reflection. If $r=r'=\bar{r}$ (case $\bar{a}$)) there is no other section; if $r<\bar{r}$ but either $r'=r$ or $r'=\bar{r}$ (case $\bar{b}$)) the section $\langle 2r, 2\bar{r}\rangle$ is a negative half wave itself. In this case if $r'=r$, then this half wave will remain negative, if $r'=r$, it will be turned into positive one. If $r<r'<\bar{r}$ (case $\bar{c}$)) the sections $\langle 2r, 2r'\rangle$ and $\langle 2r', 2\bar{r}\rangle$ are half waves themselves. The former will be turned into a positive one, the latter will remain negative. What remains to be done is to connect these half waves, namely a positive after a negative one; in case $\bar{a}$) and $\bar{c}$) beginning with a positive half wave, in case $\bar{b}$) with a negative one if $r'=r$ and with a positive one if $r'=\bar{r}$.

Each of these procedures determines uniquely the inverse construction, leading to a one-to-one mapping of the sets of corresponding paths. For the second part of this theorem similar construction can be applied.

Summation over l of the relations in Theorem 2. 3 results in the following

Corollary 2. 2. $P(s_{2r+1}=\varkappa)=P(s_{2r}=\varkappa)=P(\gamma=r)$ *for* $r=0, 1, 2, \ldots, n$.

Another fact proved herewith is expressed in the

Corollary 2. 3.

$$P(\varkappa=2l,\ \varrho=2r)=\tfrac{1}{2}P(\lambda=2l,\gamma=r)+P(\lambda=2l+1,\gamma=r,\ s_1=+1).$$

and $$P(\varkappa=2l-1,\ \varrho=2r+1)=$$

$$=P(\lambda=2l-1,\gamma=r,s_1=-1)+\tfrac{1}{2}P(\lambda=2l,\gamma=r).$$

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On regular vector measures

By N. DINCULEANU in Bucharest (Roumania)

1. Introduction

Let T be a locally compact space, $\mathcal{C}$ a nonvoid class of subsets of T, X a Banach space and $\mathbf{m}: \mathcal{C} \to X$ a set function.

A set $A \in \mathcal{C}$ is said to be *regular* (with respect to $\mathbf{m}$) if for every $\varepsilon > 0$ there exist a compact set $K \subset A$ and an open set $G \supset A$ such that if $A' \in \mathcal{C}$ and $K \subset A' \subset G$, then $|\mathbf{m}(A) - \mathbf{m}(A')| < \varepsilon$. The set function $\mathbf{m}$ is called *regular* if every set $A \in \mathcal{C}$ is regular.

The class $\mathcal{C}$ is a *lattice* if $A \in \mathcal{C}$ and $B \in \mathcal{C}$ imply $A \cup B \in \mathcal{C}$ and $A \cap B \in \mathcal{C}$.

The class $\mathcal{C}$ is a *clan* if $A \in \mathcal{C}$ and $B \in \mathcal{C}$ imply $A \cup B \in \mathcal{C}$ and $A - B \in \mathcal{C}$.

We shall denote by $\mathcal{B}$ the clan of the Borel subsets of T which are relatively compact. We call (Borel) *measure on* T with values in X, every countably additive set function defined on $\mathcal{B}$ with values in X.

By the theorem of KAKUTANI [6] a positive regular Borel measure on T can be identified with a positive Radon measure on T [1].

We shall consider sometimes the following conditions on $\mathcal{C}$:

(i) *For each compact set $K \subset T$ and each open set $G \supset K$, there exists a set $A \in \mathcal{C}$ such that $K \subset A \subset G$.*

(ii) *For each set $A \in \mathcal{C}$ there exists a set $A' \in \mathcal{C}$ such that $A \subset \operatorname{Int} A'$.*

The following result is known ([1], chap IV, § 4, No. 10, and [5], § 53, § 54).

If $\mathcal{C}$ is a lattice satisfying the condition (i), *and if a positive (finite) set function μ defined on $\mathcal{C}$ is increasing, subadditive, additive and regular, then there exists a unique positive Radon measure μ_1 on T such that the sets $A \in \mathcal{C}$ are μ_1-integrable and $\mu_1(A) = \mu(A)$ for $A \in \mathcal{C}$.*

In particular, if $\mathcal{C}$ is a *clan* satisfying the condition (i) and if the *positive* set function μ is *additive and regular*, the above conclusion remains valid (because in this case μ is also increasing and subadditive).

In this paper, we extend this last result to the case $\mathcal{C}$ is a clan satisfying the conditions (i) and (ii) and $\mathbf{m}: \mathcal{C} \to X$ is additive, regular and of finite variation (theorem 3).

In case T is a compact metric space this extension was done by C. FOIAȘ and has been exposed in [7] (Chapter 25, § 5).

In case T is compact and X is the space of the complex numbers, see [4]. We remark that in [4] the definition of the regularity is different from that used in the present paper. However, these two definitions are equivalent, for instance, if $\mathcal{C}$ contains the compact subsets of T.

2. Regular set functions

In the sequel, we shall suppose that $\mathcal{C}$ is a *clan* and that the set function $\mathbf{m}: \mathcal{C} \to X$ is *additive*.

We say that a set $A \in \mathcal{C}$ is *regular on the left (on the right)* if for every $\varepsilon > 0$ there exists a compact set $K \subset A$ (an open set $G \supset A$) such that if $A' \in \mathcal{C}$ and $K \subset A' \subset A$ $(A \subset A' \subset G)$ then $|\mathbf{m}(A) - \mathbf{m}(A')| < \varepsilon$.

All the compact sets $K \in \mathcal{C}$ are regular on the left, and all the open sets $G \in \mathcal{C}$ are regular on the right.

If all the sets $A \in \mathcal{C}$ are regular on the left (on the right) we say that the set function $\mathbf{m}$ is regular on the left (on the right).

It is clear that a regular set $A \in \mathcal{C}$ is regular on the left and on the right and we shall show that the converse is also true.

We first prove

Proposition 1. *A set $A \in \mathcal{C}$ is regular if and only if for every $\varepsilon > 0$ there exist a compact set $K \subset A$ and an open set $G \supset A$ such that if $B \in \mathcal{C}$ and $B \subset G - K$ then $|\mathbf{m}(B)| < \varepsilon$.*

A set $A \in \mathcal{C}$ is regular on the left (on the right) if and only if for every $\varepsilon > 0$ there exists a compact set $K \subset A$ (an open set $G \supset A$) such that if $B \in \mathcal{C}$ and $B \subset A - K$ $(B \subset G - A)$ then $|\mathbf{m}(B)| < \varepsilon$.

We shall prove only the part concerning the regularity. Suppose first that A is regular and let $\varepsilon > 0$. Take a compact set $K \subset A$ and an open set $G \supset A$ such that if $A' \in \mathcal{C}$ and $K \subset A' \subset G$ then $|\mathbf{m}(A) - \mathbf{m}(A')| < \frac{\varepsilon}{2}$.

If $B \in \mathcal{C}$ and $B \subset G - K$ then

$$B = (A \cup B) - (A - B), \quad A - B \subset A \cup B$$

and

$$K \subset A \cup B \subset G, \quad K \subset A - B \subset G,$$

therefore

$$|\mathbf{m}(B)| = |\mathbf{m}(A \cup B) - \mathbf{m}(A - B)| \leqq$$

$$\leqq |\mathbf{m}(A \cup B) - \mathbf{m}(A)| + |\mathbf{m}(A) - \mathbf{m}(A - B)| < \varepsilon,$$

hence A verifies the condition of the propositon.

Conversely, suppose that this condition is verified and let $\varepsilon > 0$. Take a compact set $K \subset A$ and an open set $G \supset A$ such that if $B \in \mathcal{C}$ and $B \subset G - K$ then $|\mathbf{m}(B)| < \frac{\varepsilon}{2}$.

If $A' \in \mathcal{C}$ and $K \subset A' \subset G$ then

$$A - A' \subset G - K, \quad A' - A \subset G - K$$

and

$$A' = (A \cap A') \cap (A' - A), \quad A - (A \cap A') = A - A',$$

therefore

$$|\mathbf{m}(A) - \mathbf{m}(A')| = |\mathbf{m}(A) - \mathbf{m}(A \cap A') - \mathbf{m}(A' - A)| =$$

$$= |\mathbf{m}(A - A') - \mathbf{m}(A' - A)| \leqq |\mathbf{m}(A - A')| + |\mathbf{m}(A' - A)| < \varepsilon,$$

hence A is regular.

Proposition 2. *A set $A\in\mathcal{C}$ is regular if and only if it is regular on the left and on the right.*

We have already noticed that if A is regular then it is regular on the left and on the right.

Suppose now that A is regular on the left and on the right and let $\varepsilon>0$. Take a compact set $K\subset A$ and an open set $G\supset A$ such that if $B\in\mathcal{C}$ and $B\subset A-K$ or $B\subset G-A$ then $|\mathbf{m}(B)|<\frac{\varepsilon}{2}$ (proposition 1).

If $C\in\mathcal{C}$ and $C\subset G-K$, then

$$C=(C-A)\cup(C\cap A),\quad (C-A)\cap(C\cap A)=\emptyset$$

and

$$C-A\subset G-A,\quad C\cap A\subset A-K.$$

The sets $B_1=C-A$ and $B_2=C\cap A$ are in $\mathcal{C}$ therefore $|\mathbf{m}(B_1)|<\frac{\varepsilon}{2}$ and $|\mathbf{m}(B_2)|<\frac{\varepsilon}{2}$. It follows that

$$|\mathbf{m}(C)|=|\mathbf{m}(C-A)+\mathbf{m}(C\cap A)|\leqq|\mathbf{m}(C-A)|+|\mathbf{m}(C\cap A)|<\varepsilon,$$

hence, by proposition 1, A is regular.

Remark. If $\mathbf{m}$ is not additive, it is possible that there exist sets $A\in\mathcal{C}$, regular on the left and on the right, without being regular.

Theorem 1. *Suppose that $\mathcal{C}$ verifies the condition* (ii). *Then* $\mathbf{m}$ *is regular if and only if it is regular on the left.*

By proposition 2, we have only to prove that if $\mathbf{m}$ is regular on the left then $\mathbf{m}$ is regular on the right.

Suppose that all the sets $A\in\mathcal{C}$ are regular on the left. Let $A\in\mathcal{C}$ and $\varepsilon>0$. Take a set $A'\in\mathcal{C}$ such that $A\subset\operatorname{Int}A'$. The set $A'-A$ is in $\mathcal{C}$ hence it is regular on the left: there exists a compact set $K\subset A'-A$ such that if $B\in\mathcal{C}$ and $B\subset(A'-A)-K$ then $|\mathbf{m}(B)|<\varepsilon$. Note $U=\operatorname{Int}A'$. The set $G=U-K$ is open,

$$A=U-(U-A)\subset U-K=G$$

and

$$G-A=(U-K)-A=(U-A)-K\subset(A'-A)-K,$$

therefore if $B\in\mathcal{C}$ and $B\subset G-A$, then $B\subset(A'-A)-K$ hence $|\mathbf{m}(B)|<\varepsilon$. It follows that A is regular on the right.

Remarks. 1. If all the sets $A\in\mathcal{C}$ are *relatively compact*, then condition (i) implies condition (ii).

Indeed, for every set $A\in\mathcal{C}$ we can choose a relatively compact open set $U\supset A$. If V is an arbitrary open set containing U, then by condition (i) there exists a set $A'\in\mathcal{C}$ such that $\overline{U}\subset A'\subset V$, hence $A\subset\operatorname{Int}A'$.

The condition (i) is verified, for instance, if the clan $\mathcal{C}$ contains a base of the topology of T. In particular, the condition (i) is verified if $\mathcal{C}$ contains all the compact subsets of T, or all the compact subsets of T which are G_δ ([5], § 50, theorem 4).

2. *Suppose that the sets $A \in \mathcal{C}$ are relatively compact and that $\mathcal{C}$ verifies the condition* (i). *Then* **m** *is regular if and only if it is regular on the right.*

For every set $A \in \mathcal{C}$ we take a set $A' \in \mathcal{C}$ such that $\bar{A} \subset A'$. From the right regularity of $A' - A$ we deduce that A is regular on the left.

It follows that if the sets $A \in \mathcal{C}$ are relatively compact and if $\mathcal{C}$ verifies the condition (i), then the three kinds of regularity are equivalent to each other.

3. Set functions with finite variation

For every set $A \in \mathcal{C}$ we define the variation $\mu(A)$ of **m** on A by the equality:

$$\mu(A) = \sup \sum_i |\mathbf{m}(A_i)|$$

where the supremum is taken for all the finite families (A_i) of disjoint sets $A_i \in \mathcal{C}$ contained in A.

The *variation* μ of **m** is a positive (finite or $+\infty$) and additive set function defined on $\mathcal{C}$, $\mu(\Phi) = 0$ and

$$|\mathbf{m}(A)| \leqq \mu(A) \quad \text{for} \quad A \in \mathcal{C}.$$

We say that **m** is *of finite variation* if $\mu(A) < +\infty$ for every $A \in \mathcal{C}$.

Proposition 3. *If* **m** *is of finite variation* μ, *and if* μ *is countably additive then* **m** *is countably additive.*

The proof is based on the relations

$$\left|\mathbf{m}\left(\bigcup_{i=1}^{\infty} A_i\right) - \sum_{i=1}^{n} \mathbf{m}(A_i)\right| = |\mathbf{m}(\bigcup_{i>n} A_i)| \leqq \mu(\bigcup_{i>n} A_i) = \sum_{i>n} \mu(A_i)$$

where (A_i) is a sequence of disjoint sets in $\mathcal{C}$ with $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$.

Proposition 4. *Suppose that* **m** *is of finite variation*. Then **m** *is regular on the left if and only if its variation* μ *is regular on the left.*

If μ is regular on the left, we deduce immediately that **m** is regular on the left, using the inequality $|\mathbf{m}(B)| \leqq \mu(B)$ and proposition 2.

Conversely, suppose that **m** is regular on the left. Let $A \in \mathcal{C}$ and $\varepsilon > 0$. Let $(A_i)_{1 \leqq i \leqq n}$ be a finite family of disjoint sets $A_i \in \mathcal{C}$, contained in A, such that

$$\mu(A) < \sum_{i=1}^{n} |\mathbf{m}(A_i)| + \frac{\varepsilon}{2}.$$

Because each set A_i is regular on the left (with respect to **m**) there exists a compact set $K_i \subset A_i$, such that if $A_i' \in \mathcal{C}$ and $K_i \subset A_i' \subset A_i$ then $|\mathbf{m}(A_i) - \mathbf{m}(A_i')| < \frac{\varepsilon}{2n}$.

The set $K = \bigcup_{i=1}^{n} K_i$ is compact and $K \subset A$. Let now $A' \in \mathcal{C}$ be such that $K \subset A' \subset A$. For each i, the set $A_i' = A' \cap A_i$ is in $\mathcal{C}$ and $K_i \subset A_i' \subset A_i$; the sets A_i' are disjoint, therefore

$$\mu(A') \geqq \sum_{i=1}^{n} |\mathbf{m}(A_i')|$$

hence

$$0 \leqq \mu(A) - \mu(A') \leqq \sum_{i=1}^{n} |\mathbf{m}(A_i)| + \frac{\varepsilon}{2} - \sum_{i=1}^{n} |\mathbf{m}(A_i')| =$$

$$= \sum_{i=1}^{n} [|\mathbf{m}(A_i)| - |\mathbf{m}(A_i')|] + \frac{\varepsilon}{2} \leqq \sum_{i=1}^{n} |\mathbf{m}(A_i) - \mathbf{m}(A_i')| + \frac{\varepsilon}{2} < \varepsilon.$$

It follows that A is regular on the left with respect to μ, hence μ is regular on the left.

Theorem 2. *Suppose that $\mathcal{C}$ verifies the condition* (ii) *and that* $\mathbf{m}$ *is of finite variation. Then* $\mathbf{m}$ *is regular if and only if its variation μ is regular.*

Using the inequality $|\mathbf{m}(B)| \leqq \mu(B)$ and the proposition 2, we see that if μ is regular, then $\mathbf{m}$ is regular.

Conversely, if $\mathbf{m}$ is regular, then $\mathbf{m}$ is regular on the left; by proposition 4, μ is regular on the left and by theorem 1, μ is regular.

Remark. The conclusion of the theorem 2 remains valid if $\mathcal{C}$ is a clan of relatively compact sets and if $\mathcal{C}$ verifies the conditions (i). In particular, we have

Corollary. *If* $\mathbf{m}$ *is a regular Borel measure with finite variation, then its variation is a positive regular Borel measure.*

Indeed, the measure $\mathbf{m}$ is defined on the clan $\mathcal{B}$ of the relatively compact Borel sets, which verifies the condition (ii).

4. Extension of a regular additive set function to a measure

Proposition 5. *Let μ be a positive Radon measure on T and suppose that the sets of $\mathcal{C}$ are μ-integrable and that $\mathcal{C}$ verifies the condition* (i)*. Then for every μ-integrable set $E \subset T$ and every number $\varepsilon > 0$, there exists a set $A \in \mathcal{C}$ such that $\mu(E \Delta A) < \varepsilon$.*

Let $E \subset T$ be a μ-integrable set and let $\varepsilon > 0$.

There exist a compact set $K \subset E$ and a μ-integrable open set $G \supset E$ such that $\mu(G - K) < \frac{\varepsilon}{2}$ ([1], chap IV, § 4, no 6, theorem 4). Because $\mathcal{C}$ verifies the condition (i), there exists a set $A \in \mathcal{C}$ such that $K \subset A \subset G$. Then the sets $E - A$ and $A - E$ are μ-integrable and contained in $G - K$, therefore

$$\mu(E \Delta A) = \mu\big((E - A) \cup (A - E)\big) = \mu(E - A) + \mu(A - E) \leqq 2\mu(G - K) < \varepsilon.$$

Proposition 6. *Let μ be a positive Radon measure on T and suppose that the sets of $\mathcal{C}$ are μ-integrable and that $\mathcal{C}$ verifies the condition* (i)*. Then the set $\mathcal{E}(\mathcal{C})$ of the step functions of the form $\sum_{i=1}^{n} \varphi_{A_i} \alpha_i$ with $A_i \in \mathcal{C}$, is dense in $\mathfrak{L}^1(\mu)$.*

We know that the set $\mathscr{E}(\mathscr{B})$ of the step functions of the form $\sum_{i=1}^{n} \varphi_{E_i}\alpha_i$, where E_i are relatively compact Borel sets, is dense in $\mathfrak{L}^1(\mu)$. Then it is sufficient to prove that $\mathscr{E}(\mathcal{C})$ is dense in $\mathscr{E}(\mathscr{B})$ for the topology of $\mathfrak{L}^1(\mu)$.

Let $f=\sum_{i=1}^{n} \varphi_{E_i}\alpha_i$ be a function in $\mathscr{E}(\mathscr{B})$ with $\alpha_i \neq 0$ for every i, and let $\varepsilon > 0$. For each i there exists a set $A_i \in \mathcal{C}$ such that

$$\mu(E_i \Delta A_i) < \frac{\varepsilon}{n|\alpha_i|}.$$

The function $g = \sum_{i=1}^{n} \varphi_{A_i}\alpha_i$ is in $\mathscr{E}(\mathcal{C})$ and

$$|f-g| = \left|\sum_{i=1}^{n} (\varphi_{E_i} - \varphi_{A_i})\alpha_i\right| \leqq \sum_{i=1}^{n} |\varphi_{E_i} - \varphi_{A_i}|\,|\alpha_i| = \sum_{i=1}^{n} \varphi_{E_i \Delta A_i}\alpha_i,$$

therefore $\int |f-g|\, d\mu \leqq \sum_{i=1}^{n} \mu(E_i \Delta A_i)|\alpha_i| < \varepsilon$.

It follows that $\mathscr{E}(\mathcal{C})$ is dense in $\mathscr{E}(\mathscr{B})$, hence $\mathscr{E}(\mathcal{C})$ is dense in $\mathfrak{L}^1(\mu)$.

Remark. The propositions 5 and 6 are valid for an arbitrary class $\mathcal{C}$ verifying the condition (i). We can take $\mathcal{C}$ to be, for instance: the class of the compact sets; the class of the compact sets which are G_δ ([5], § 50, theorem 4); the class of the relatively compact open sets; the class of the relatively compact open sets which are F_σ ([5], § 50, theorem 4).

If $\mathbf{m}_1 : \mathscr{B} \to X$ is a regular Borel measure with finite variation μ_1, the μ_1-integrable real functions are called $\mathbf{m}_1$-integrable and we put $\mathfrak{L}^1(\mathbf{m}_1) = \mathfrak{L}^1(\mu_1)$. For every $\mathbf{m}_1$-integrable function $f \in \mathfrak{L}^1(\mathbf{m}_1)$ it is defined the integral $\int f d\mathbf{m}_1$ (see [2], [3] and [7]).

For every $\mathbf{m}_1$-integrable set $A \subset T$ (with $\varphi_A \in \mathfrak{L}^1(\mathbf{m}_2)$) we put $\mathbf{m}_1(A) = \int \varphi_A d\mathbf{m}_1$.

Theorem 3. *Suppose that $\mathcal{C}$ verifies the conditions* (i) *and* (ii) *and that* $\mathbf{m}$ *is regular with finite variation μ. Then there exists a unique regular Borel measure* $\mathbf{m}_1$ *with finite variation μ_1 such that the sets $A \in \mathcal{C}$ are $\mathbf{m}_1$-integrable and* $\mathbf{m}_1(A) = \mathbf{m}(A)$ *for $A \in \mathcal{C}$. In this case we have $\mu_1(A) = \mu(A)$ for $A \in \mathcal{C}$.*

The variation μ of $\mathbf{m}$ is a positive and additive set function defined on the clan $\mathcal{C}$. Because $\mathcal{C}$ verifies the condition (ii), μ is regular (theorem 2). Because $\mathcal{C}$ verifies the condition (i), there exists a positive Radon measure ν on T such that the sets $A \in \mathcal{C}$ are ν-integrable and $\nu(A) = \mu(A)$ for $A \in \mathcal{C}$ (see Introduction). Then $|\mathbf{m}(A)| \leqq \nu(A)$ for $A \in \mathcal{C}$. For each step function $f = \sum_i \varphi_{A_i}\alpha_i \in \mathscr{E}(\mathcal{C})$ put

$$U(f) = \sum_i \mathbf{m}(A_i)\alpha_i.$$

The definition of $U(f)$ is independent of the form in which f can be written as a step function.

The mapping $f \to U(f)$ of $\mathscr{E}(\mathcal{C})$ in X is linear; it is also continuous for the topology of $\mathfrak{L}^1(\nu)$, because, taking the sets A_i disjoint, we have

$$\|U(f)\|_1 \leqq \sum_i |\mathbf{m}(A_i)|\,|\alpha_i| \leqq \sum_i \nu(A_i)|\alpha_i| = \int |f|\,d\nu.$$

Because $\mathcal{C}$ verifies the condition (i), $\mathscr{E}(\mathcal{C})$ is dense in $\mathfrak{L}^1(\nu)$ (proposition 6), therefore U can be uniquely extended to a continuous linear mapping $U_1 : \mathfrak{L}^1(\nu) \to X$ and we have

$$\|U_1(f)\|_1 \leqq \int |f|\,d\nu \quad \text{for} \quad f \in \mathfrak{L}^1(\nu).$$

For every relatively compact Borel set $A \in \mathscr{B}$ put

$$\mathbf{m}_1(A) = U_1(\varphi_A).$$

It is clear that $\mathbf{m}_1$ is additive on $\mathscr{B}$ and that

$$|\mathbf{m}_1(A)| \leqq \nu(A) \text{ for } A \in \mathscr{B}.$$

From this inequality we deduce that $\mathbf{m}_1$ is countably additive, regular and of finite variation μ_1, therefore $\mathbf{m}_1$ *is a regular Borel measure with finite variation.* It follows that μ_1 is a positive regular Borel measure and that

$$\mu_1(A) \leqq \nu(A) \text{ for } A \in \mathscr{B}$$

i. e. $\mu_1 \leqq \nu$. Then $\mathfrak{L}^1(\nu) \subset \mathfrak{L}^1(\mu_1) = \mathfrak{L}^1(\mathbf{m}_1)$. Because the sets $A \in \mathcal{C}$ are ν-integrable, we deduce that these sets are $\mathbf{m}_1$*-integrable.*

For every function $f \in \mathfrak{L}^1(\nu)$ we have

$$\left|\int f\,d\mathbf{m}_1\right| \leqq \int |f|\,d\mu_1 \leqq \int |f|\,d\nu$$

therefore the linear mapping $f \to \int f d\mathbf{m}_1$ of $\mathfrak{L}^1(\nu)$ into X is continuous.

On the other hand, for the step functions $f = \sum_i \varphi_{A_i}\alpha_i \in \mathscr{E}(\mathscr{B})$ we have

$$\int f\,d\mathbf{m}_1 = \sum_i \mathbf{m}_1(A_i)\alpha_i = U_1(f).$$

Because $\mathscr{E}(\mathscr{B})$ is dense in $\mathfrak{L}^1(\nu)$ and the continuous linear mappings $f \to \int f d\mathbf{m}_1$ and $U_1(f)$ of $\mathfrak{L}^1(\nu)$ into X coincide on $\mathscr{E}(\mathscr{B})$, we deduce that

$$\int f\,d\mathbf{m}_1 = U_1(f), \quad \text{for every} \quad f \in \mathfrak{L}^1(\nu).$$

In particular, for every set $A \in \mathcal{C}$ we have

$$\mathbf{m}_1(A) = \int \varphi_A\,d\mathbf{m}_1 = U_1(\varphi_A) = U(\varphi_A) = \mathbf{m}(A).$$

Because $\mu_1 \leqq \nu$, we have $\mu_1(A) \leqq \nu(A)$ for every ν-integrable set $A \subset T$. In particular, we have

$$\mu_1(A) \leqq \nu(A) = \mu(A) \text{ for } A \in \mathcal{C}.$$

Conversely, if $A \in \mathcal{C}$ and if (A_i) is a finite family of disjoint sets of $\mathcal{C}$ contained in A, we have

$$\sum_i |\mathbf{m}(A_i)| = \sum_i |\mathbf{m}_1(A_i)| \leqq \sum_i \mu_1(A_i) = \mu_1(\bigcup_i A_i) \leqq \mu_1(A)$$

hence $\mu(A) \leqq \mu_1(A)$, therefore

$$\mu_1(A) = \mu(A) \quad \text{for} \quad A \in \mathcal{C}.$$

From the uniqueness of ν we deduce that $\mu_1 = \nu$. Let now $\mathbf{m}_2$ be a regular Borel measure with finite variation μ_2 such that the sets of $\mathcal{C}$ are $\mathbf{m}_2$-integrable and $\mathbf{m}_2(A) = \mathbf{m}(A)$ for $A \in \mathcal{C}$.

We have then $\mathbf{m}_1(A) = \mathbf{m}_2(A)$ for $A \in \mathcal{C}$, therefore

$$\int f\, d\mathbf{m}_1 = \int f\, d\mathbf{m}_2 \quad \text{for} \quad f \in \mathcal{E}(\mathcal{C}).$$

By proposition 6, the set $\mathcal{E}(\mathcal{C})$ is dense in the space $\mathfrak{L}^1(\mu_1 + \mu_2)$. Because $\mathfrak{L}^1(\mu_1 + \mu_2)$ is contained in $\mathfrak{L}^1(\mu_1)$ and in $\mathfrak{L}^1(\mu_2)$, the linear mappings $f \to \int f d\mathbf{m}_1$ and $f \to \int f d\mathbf{m}_2$ are defined and continuous on $\mathfrak{L}^1(\mu_1 + \mu_2)$ and coincide on the dense set $\mathcal{E}(\mathcal{C})$, therefore

$$\int f\, d\mathbf{m}_1 = \int f\, d\mathbf{m}_2 \quad \text{for} \quad f \in \mathfrak{L}^1(\mu_1 + \mu_2),$$

In particular

$$\mathbf{m}_1(A) = \mathbf{m}_2(A) \quad \text{for} \quad A \in \mathcal{B},$$

hence $\mathbf{m}_1 = \mathbf{m}_2$. This proves the uniqueness of $\mathbf{m}_1$ and completes the proof of the theorem.

Corollary 1. *If $\mathcal{C}$ is a clan verifying the conditions* (i) *and* (ii), *every additive and regular set function* $\mathbf{m}: \mathcal{C} \to X$ *with finite variation is countably additive.*

In particular, every additive and regular set function $\mathbf{m}: \mathcal{B} \to X$ with finite variation is a regular Borel measure with finite variation.

Corollary 2. *If $\mathcal{C}$ is a clan contained in $\mathcal{B}$ and containing a base of the topology of T, then every additive and regular set function* $\mathbf{m}: \mathcal{C} \to X$ *with finite variation can be extended to a regular Borel measure with finite variation.*

In particular, corollaries 1 and 2 are valid in each of the following cases: $\mathcal{C}$ is the clan generated by all the compact sets; $\mathcal{C}$ is the clan generated by all the compact sets which are G_δ; $\mathcal{C}$ is the clan of the Baire sets wich are relatively compact; T is totally disconnected and $\mathcal{C}$ is the clan of all compact-open subsets of T.

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О ПРОИЗВЕДЕНИЯХ УПОРЯДОЧЕННЫХ АВТОМАТОВ. I

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В настоящей статье исследуется вопрос о вложимости конечного множества частично упорядоченных автоматов в произведение автоматов в смысле В. М. Глушкова. Полученный результат обобщает одну теорему Й. Хартманиса, относяшуюся к некоторому частному виду произведения автоматов.

Относительно терминологии мы ссылаемся на работу [1] Глушкова. Автомат $\mathbf{A}(A, X, Y, \delta, \lambda)$ мы назовем упорядоченным, если множества его входных сигналов, выходных сигналов и состояний — упорядочены, и выполняются следующие условия:

a) если $x_1 \geqq x_2$, то $\delta(a, x_1) \geqq \delta(a, x_2)$; $x_1, x_2 \in X, a \in A$ произвольные,

b) если $a_1 \geqq a_2$, то $\delta(a_1, x) \geqq \delta(a_2, x)$; $a_1, a_2 \in A, x \in X$ произвольные,

c) если $a_1 \geqq a_2$, то $\lambda(a_1) \geqq \lambda(a_2)$.

В дальнейшем, говоря об автомате, мы всегда будем подразумевать упорядоченный автомат.

Пусть дано множество произвольных конечных автоматов $\mathbf{A}_i(A_i, X_i, Y_i, \delta_i, \lambda_i)$ $(i=1, 2, \dots, k)$. В соглащение с терминологией Глушкова произведением этих автоматов мы называем автомат $\mathbf{A}$, с множеством состояний $A = A_1 \times A_2 \times \dots \times A_k$, где отношение упорядоченности на множестве A задается следующим образом:

$$a[=(a_1, a_2, \dots, a_k)] \geqq a'[=(a_1', a_2', \dots, a_k')],$$

если для всех пар a_i, a_i' имеет место неравенство $a_i \geqq a_i'$. Упорядочение входных сигналов в $\mathbf{A}$ — произвольно. Функция обратной связи $\varphi: A \times X \to X_1 \times \dots \times X_k$ подчиняется следующим условиям:

1) если $x \geqq x' (\in X)$, то $\varphi(a_1, \dots, a_k, x) \geqq \varphi(a_1, \dots, a_k, x')$,

2) если $(a_1, \dots, a_k) \geqq (a_1', \dots, a_k')$, то $\varphi(a_1, \dots, a_k, x) \geqq \varphi(a_1', \dots, a_k', x)$.

($X_1 \times \dots \times X_k$ упорядочено аналогично множеству $A_1 \times \dots \times A_k$). Функция δ перехода автомата $\mathbf{A}$ определена обычным путем: $\delta(a, x) = (\delta_1(a_1, x_1), \dots, \delta_k(a_k, x_k))$, где $a = (a_1, \dots, a_k)$ и $(x_1, \dots, x_k) = \varphi(a_1, \dots, a_k, x)$. Наконец, упорядочение множества Y выходных сигналов автомата $\mathbf{A}$ — такое, что $a \geqq a'$ влечет $\lambda(a) \geqq \lambda(a')$.

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Будем пользоваться следующими обозначениями: если π — произвольное разбиение множества M, $a, a' \in M$ и a, a' содержится в одном и том же классе по разбиению π, то мы пишем: $a' \equiv a(\pi)$. Допустимым разбиением произвольного автомата мы называем разбиение этого автомата, при котором из $a \equiv a'(\pi)$ следует $\delta(a, x) \equiv \delta(a', x)(\pi)$ и если $a \leqq b$, $a \equiv a'(\pi)$, $b \equiv b'(\pi)$, то не может выполняться $a' > b'$. В этом случае мы пишем $\pi(a) \leqq \pi(b)$, где $\pi(a)$ — множество всех таких элементов a', для которых имеет место $a \equiv a'(\pi)$.

Рассмотрим конечное множество автоматов $\mathbf{A}_j (j=1, \ldots, k)$ с отношением частичной упорядоченности R. Обозначим через $P(\mathbf{A}_i)$ множество, состоящее из всех автоматов из множества $\{\mathbf{A}_j\}$, которые больше автомата $\mathbf{A}_i$, и из самого автомата $\mathbf{A}_i$. Произведение $\mathbf{A}$ автоматов $\mathbf{A}_j$ мы назовем их R-произведением, если для функции φ обратной связи выполняются 1), 2) и

3) если $\varphi(a_1, \ldots, a_k, x) = (x_1, \ldots, x_k)$, то $x_i = \varphi_i(a_{i_1}, a_{i_2}, \ldots, a_{i_m}, a_i, x)$, где $\mathbf{A}_{i_1}, \ldots, \mathbf{A}_{i_m}$ — все автоматы, которые больше автомата $\mathbf{A}_i$.

Рассмотрим конечные множества $\{\mathbf{M}_i\}$ $(i=1, 2, \ldots, n)$ соотв. $\{\mathbf{A}_j\}$ $(j=1, 2, \ldots, k)$ произвольных автоматов. Мы говорим, что множество $\{\mathbf{M}_i\}$ представимо множеством $\{\mathbf{A}_j\}$, если существует такое R-произведение автоматов $\mathbf{A}_j$, которое для всех i содержит в качестве подавтомата автомат, изоморфный автомату $\mathbf{M}_i$.

Имеет место

Теорема 1. Если множество $M = \{\mathbf{M}_r\}$ $(r=1, \ldots, n)$ представляется R-произведением $\mathbf{A}$ автоматов $\mathbf{A}_i$ $(i=1, \ldots, k)$, то каждое множество $P(\mathbf{A}_i)$ индуцирует допустимое разбиение π_{r_i} каждого автомата $\mathbf{M}_r$, где состояния из $\mathbf{M}_r$ тогда и только тогда содержатся в одном и том же класса разбиения π_{r_i}, если компоненты их образов в данном изоморфизме вложения $\mathbf{M}_r$ в $\mathbf{A}$, взятые из любого $\mathbf{A}_j \in P(\mathbf{A}_i)$, совпадают. Если $\mathbf{A}_i \geqq \mathbf{A}_j$, то разбиение с индуцирующим множеством $P(\mathbf{A}_i)$ больше разбиения, индуцирующегося множеством $P(\mathbf{A}_j)$. Пересечение всех таких разбиений в $\mathbf{M}_r$ является тривиальным.

Обратно, пусть даны допустимые разбиения $\pi_{r_1}, \ldots, \pi_{r_{i_r}}$ в автоматах $\mathbf{M}_r$ $(r=1, \ldots, n)$, пересечение которых является тривиальным. Рассмотрим множество разбиений $\pi_{r_1}, \ldots, \pi_{r_{i_r}}, \pi_{r_0}$, дге π_{r_0} — разбиение автомата $\mathbf{M}_r$, для которого имеет место $m \equiv m'(\pi_{r_0})$ для всех $m, m' \in \mathbf{M}_r$. Тогда существуют конечное множество автоматов $\{A_i\}$ $(i=1. \ldots, k)$ и упорядочение R этого множества, такие, что каждый $\mathbf{M}_r$ представляется некоторым R-произведением автоматов $\mathbf{A}_i$, причем каждому разбиению π_{r_i} соответствует некоторое $\mathbf{A}_i$ так, что разбиение в $\mathbf{M}_r$, индуцирующееся множеством $P(\mathbf{A}_i)$, совпадает с π_{r_i}. Далее, каждое $P(\mathbf{A}_i)$ индуцирует некоторое π_{r_i} во всех автоматах $\mathbf{M}_r$. $\pi_{r_i} \geqq \pi_{r_j}$, если для индуцирующих $P(\mathbf{A}_i)$ и $P(\mathbf{A}_j)$ имеет место $\mathbf{A}_i \geqq \mathbf{A}_j$.

Доказательство. Пусть дано множество автоматов $\{\mathbf{M}_r\}$ $(r=1, \ldots, n)$ и множество автоматов $\{\mathbf{A}_i\}$ $(i=1, \ldots, k)$ с (частичным) упорядочением R. Рассмотрим R-произведение $\mathbf{A}$ автоматов $\mathbf{A}_i$, в котором все автоматы $\mathbf{M}_r$ изоморфно вложены. Берем такие разбиения π_i автомата $\mathbf{M}_r$, что $m_r \equiv m_r'(\pi_i)$ $(m_r, m_r' \in \mathbf{M}_r)$, если компоненты их образов в $\mathbf{A}$, содержащиеся в $\mathbf{A}_j$ $(\mathbf{A}_j \in P(\mathbf{A}_i))$, совпадают. Покажем, что эти разбиения допустимы. С одной стороны, если $m_r \equiv m_r'(\pi_i)$, то $\delta_r(m_r, x_r) \equiv \delta_r(m_r', x_r)(\pi_i)$ $(x_r \in X_r)$. Действительно, пусть χ_r изоморфизм автомата $\mathbf{M}_r$ в $\mathbf{A}$:

$$\chi_r(m_r) = (\ldots, a_j, \ldots, a_k, \ldots) \quad (a_j \in A_j,\ \mathbf{A}_j \in P(\mathbf{A}_i)).$$

$$\chi_r(m_r') = (\ldots, a_j, \ldots, a_k', \ldots) \quad (a_k, a_k' \in A_k,\ \mathbf{A}_k \notin P(\mathbf{A}_i)).$$

Тогда

$$\delta(\ldots, a_j, \ldots, a_k, \chi_r(x_r)) = (\ldots, \delta_j(a_j, x_j), \ldots, \delta_k(a_k, x_k), \ldots),$$

$$\delta(\ldots, a_j, \ldots, a_k', \chi_r(x_r)) = (\ldots, \delta_j(a_j, x_j'), \ldots, \delta_k(a_k', x_k'), \ldots),$$

где

$$(\ldots, x_j, \ldots, x_k, \ldots) = \varphi(\ldots, a_j, \ldots, a_k, \ldots, \chi_r(x_r)),$$

$$(\ldots, x_j', \ldots, x_k', \ldots) = \varphi(\ldots, a_j, \ldots, a_k', \ldots, \chi_r(x_r)).$$

Но, по определению R-произведения, $x_j = x_j'$.

Если класс C_i по разбиению π_i определяется элементами a_j $(\in A_j, \mathbf{A}_j \in P(\mathbf{A}_i))$, и класс C_i' элементами a_j^*, то имеет место один из следующих утверждений:

α) можно найти такое j $(\mathbf{A}_j \in P(\mathbf{A}_i))$, что элемент a_j не находится в отношении упорядоченности с элементом a_j^*.

Если все a_j $(\in A_j, \mathbf{A}_j \in P(\mathbf{A}_i))$ находятся в отношении упорядоченности с соответствующими элементами a_j^*, то

β) найдутся такие j и l, $(\mathbf{A}_j, \mathbf{A}_l \in P(\mathbf{A}_i))$, что $a_j > a_j^*$, $a_l < a_l^*$;

γ) для всех j $(\mathbf{A}_j \in P(\mathbf{A}_i))$: $a_j \geqq a_j^*$;

δ) для всех j $(\mathbf{A}_j \in P(\mathbf{A}_i))$: $a_j \leqq a_j^*$.

Очевидно, в случаях α) и β) ни один элемент из C_i не находится в отношении упорядоченности ни с одним элементом из C_i'.

В случае γ) соотв. δ) произвольный элемент из класса C_i не меньше (соотв. не больше) любого элемента из класса C_i'.

Этим доказано, что разбиения π_i — допустимы.

Пусть теперь $\mathbf{A}_i \leqq \mathbf{A}_j$. Тогда $m_r \equiv m_r'(\pi_i)$ влечет $m_r \equiv m_r'(\pi_j)$, т. е. $\pi_i \leqq \pi_j$.

Остается показать, что пересечение π этих допустимых разбиений является тривиальным. Предположим, что π обладает таким классом, который содержит два различных состояния m_r и m_r'. Но в этом случае $\chi_r(m_r)=\chi_r(m_r')$, вопреки предположению, согласно которому χ_r — изоморфизмы. Этим первая часть теоремы доказана.

Для дальнейших предпосылаем несколько определений. Мы говорим, что состояние a автомата $\mathbf{A}$ — изолированное, если любое из $a\leqq b$ и $a\geqq b$ влечет за собой $b=a$.

Рассмотрим множество M_i $(i=1, \dots, k)$ и некоторое его частичное упорядочение R. Мы говорим, что M_i опередит M_j, если $M_i>M_j$, но ни с каким $M_k(\in\{M_i\})$ не выполняется $M_i>M_k>M_j$.

Теперь рассмотрим конечное множество M таких автоматов $\mathbf{M}_r$ $(r=1, \dots, n)$, что каждый автомат $\mathbf{M}_r$ из M обладает разбиениями $\pi_{r_1}, \dots, \pi_{r_{i_r}}$, пересечение которых является тривиальным.

Покажем, что можно найти конечное множество $\{\mathbf{A}_i\}$ $(i=1, \dots, k)$ таких автоматов и такое упорядочение R этого множества, чтобы M представлялся некоторым R-произведением автоматов $\mathbf{A}_i$.

Рассмотрим произвольный автомат $\mathbf{M}_r$ (из M) и некоторое допустимое разбиение π_r этого автомата. Тогда можно конструировать автоматы $\mathbf{M}_{\pi_r}$, множество входных сигналов которых есть $(\pi_{r_i}(m_{r_i}), \dots, \pi_{r_j}(m_{r_j}), x_r)$, где $\pi_{r_i}, \dots, \pi_{r_j}$ — все допустимые разбиения, опережающие разбиение π_r, а $x_r(\in X_r)$ — произвольно. Множествами состояний (и выходных сигналов) служат классы автомата $\mathbf{M}_r$ по разбиению π_r. Функция перехода автомата $\mathbf{M}_{\pi_r}$ определяется следующим образом:

Обозначим через $T=\{\pi_{r_1}, \dots, \pi_{r_{i_r}}\}$ множество заданных разбиений автомата $\mathbf{M}_r$. Далее, пусть T_1 — множество максимальных элементов множества T. Если $\pi_{r_k}\in T_1$, то $\delta_{r_k}(\pi_{r_k}(m_k), x_r)=\pi_{r_k}(\delta(m_k, x_r))$. Пусть T_2 — множество максимальных элементов множества $T-T_1$, и $\pi_{r_j}\in T_2$. Если теперь $\pi_{r_j}(m)$ — изолированное и существуют такие классы $\pi_{r_k}(m_k'), \dots$ (по всеми разбиениями, опережающими разбиение π_{r_j}), что $\delta_{r_k}(\pi_{r_k}(m_k', x_r))=\pi_{r_k}(m_k), \dots$ и для всех таких классов $\bigcap\pi_{r_k}(m_k')=a_k\neq\varnothing$, $\pi_{r_k}(m_k'), \pi_{r_k}(m_k), \dots$ — изолированные и $\pi_{r_j}(m)\subsetneqq a_k$,[1] то мы выберем произвольный класс $\pi_{r_j}(m')$ из некоторого a_k и положим в этом случае

$$\delta_{r_j}(\pi_{r_j}(m), (\pi_{r_k}(m_k), \dots, x_r))=\pi_{r_j}(\delta_r(m', x_r)),$$

[1]) Если подставляя x_r' в место x_r получаются эти же соотношения, то пусть

$$\delta_{r_j}(\pi_{r_j}(m), (\pi_{r_k}(m_k), \dots, x_r'))=\pi_{r_j}(\delta_r(m', x_r'))$$

соотв.

$$\delta_{r_i}(\pi_{r_i}(m_i), (\pi_{r_j}(m_j), \dots, x_r'))=\pi_{r_i}(\delta_r(m_i', x_r')).$$

а во всех остальных случаях

$$\delta_{r_j}(\pi_{r_j}(m), (\pi_{r_k}(m_k), \ldots, x_r)) = \pi_{r_j}(\delta_r(m, x_r)).$$

Теперь обозначим через T_3 множество максимальных элементов множества $T-(T_1+T_2)$. Пусть $\pi_{r_i}(\in T_3)$ — произвольно. Если $\pi_{r_i}(m_i)$ — изолированный и существуют такие классы $\pi_{r_j}(m_j')$, ... (по всеми разбиениями, опережающими разбиение π_{r_i}), что

$$\delta_{r_j}(\pi_{r_j}(m_j'), (\pi_{r_k}(m_k'), \ldots, x_r)) = \pi_{r_j}(m_j), \ldots$$

и для всех таких классов $\bigcap \pi_{r_j}(m_j') = a_j \neq \emptyset$, $\pi_{r_j}(m_j')$, $\pi_{r_j}(m_j)$, ... — изолированные и $\pi_{r_i}(m_i) \subseteq a_j$, (см. сноску[1]), то мы выберем произвольный класс $\pi_{r_i}(m_i')$ из некоторого a_j. В этом случае пусть

$$\delta_{r_i}(\pi_{r_i}(m_i), (\pi_{r_j}(m_j), \ldots, x_r)) = \pi_{r_i}(\delta_r(m_i', x_r))$$

и

$$\delta_{r_i}(\pi_{r_i}(m_i), (\pi_{r_j}(m_j), \ldots, x_r)) = \pi_{r_i}(\delta_r(m_i, x_r))$$

во всех остальных случаях.

Продолжая этот процесс, мы конструируем автоматы для всех разбиений π_{r_k}. Упорядочение множества входных сигналов автомата $\mathbf{M}_{\pi_r}$ мы определим так: $(\pi_{r_i}(m_{r_i}), \ldots, x_r) \geqq (\pi_{r_i}(m'_{r_i}), \ldots, x'_r)$, тогда и только тогда, если $\pi_{r_i}(m_{r_i}) \geqq \pi_{r_i}(m'_{r_i})$ и $x_r \geqq x'_r$. Упорядочение множества состояний соотв. выходных сигналов определяется естественным образом.

Множество входных сигналов соотв. состояний мы дополним с входным сигналом x_{π_r} соотв. состоянием a_{π_r}, для которых $\delta_{\pi_r}(\pi_r(m_r), x_{\pi_r}) = \pi_r(m_r)$ и $\delta_{\pi_r}(a_{\pi_r}, x) = a_{\pi_r}$, где $(m_r \in \mathbf{M}_r)$ состояние $\pi_r(m_r)$ и входный сигнал x автомата $\mathbf{M}_{\pi_r}$ — произвольны.

Рассмотрим множество M' автоматов $\mathbf{M}_{\pi_r}$, для всех r. Мы вводим в M' отношение частичной упорядоченности R: $\mathbf{M}_{\pi_r} \geqq \mathbf{M}_{\pi'_r}$ тогда и только тогда, если $\pi_r \geqq \pi'_r$. Сконструируем R-произведение $\mathbf{A}$ этих автоматов: множеством входных сигналов автомата $\mathbf{A}$ является объединение множеств входных сигналов автоматов $\mathbf{M}_{\pi_r}$, а множеством выходных сигналов — обединение множеств выходных сигналов автоматов $\mathbf{M}_{\pi_r}$ с множеством N всех состояний из $\mathbf{A}$, содержащих такие компоненты $\pi_{r_1}(m_1), \ldots, \pi_{r_{i_r}}(m_{i_r})$, для которых $\pi_{r_1}(m_1) \cap \ldots \cap \pi_{r_{i_r}}(m_{i_r}) = \emptyset$, или такой компонент $\pi_n(m_n)$ $(n \neq r_j;\ j=1, \ldots, i_r)$, что $\pi_n(m_n) \neq a_{\pi_n}$.

Функция φ обратной связи автомата $\mathbf{A}$ — следующая:

$$\varphi(a, x_r) = \varphi(\ldots, \pi_{1_l}(m_{1_l}), \ldots; \ldots, \pi_{r_s}(m_{r_s}) \ldots; \ldots, \pi_{n_t}(m_{n_t}), \ldots; x_r) =$$
$$= (\ldots, x_{\pi_{1l}}, \ldots; \ldots; \ldots, \varphi_{r_s}(\pi_{j_k}(m_{j_k}), \ldots, x_r), \ldots; \ldots, x_{\pi_{n_t}}, \ldots),$$

где $\pi_{j_k}(m_{j_k})$ — произвольные состояния автоматов $\mathbf{M}_{\pi_{j_k}}$, большего автомата $\mathbf{M}_{\pi_{r_s}}$, а функции φ_i определены следующим образом:

Если $\pi_{r_k}(m_{r_k})$, ... — компоненты состояния a и $\pi_{r_k}\in T_1$, то $\varphi_{r_k}(x_r) = x_r$. Если $\pi_{r_j}(m_{r_j})$, ... — компоненты состояния a ($\pi_{r_j}\in T_2$), опережены классами в a $\pi_{r_{ki}}(m_{r_{ki}})$, ..., то

$$\varphi_{r_j}(\pi_{r_{ki}}(m_{r_{ki}}), \ldots, x_r) = (\delta_{r_{ki}}(\pi_{r_{ki}}(m_{r_{ki}}), x_r), \ldots, x_r).$$

Если $\pi_{r_s}(m_{r_s})$, ... — компоненты состояния a ($\pi_{r_s}\in T_3$), опережены классами в a $\pi_{r_{jl}}(m_{r_{jl}})$, ..., то

$$\varphi_{r_s}(\pi_{r_{jl}}(m_{r_{jl}}), \ldots; \ldots, \pi_{r_{ki_l}}(m_{r_{ki_l}}), \ldots; x_r) =$$
$$= \delta_{r_{jl}}(\pi_{r_{jl}}(m_{r_{jl}}), (\delta_{r_{ki_l}}(\pi_{r_{ki_l}}(m_{r_{ki_l}}), x_r), \ldots, x_r)),$$

и т. д.

Функцию выходов λ автомата **A** мы определим так:

$$\lambda(a) = \lambda(\ldots, \pi_{1_l}(m_{1_l}), \ldots; \ldots, \pi_{r_s}(m_{r_s}), \ldots; \ldots, \pi_{n_t}(m_{n_t})) = y_r$$

если $\pi_{1_l}(m_{1_l}) = a_{\pi_{1_l}}, \pi_{r_{s_1}}(m_{r_{s_1}}) \cap \ldots \cap \pi_{r_{s_k}}(m_{r_{s_k}}) = m_r$ и $\lambda_r(m_r) = y_r$, $\pi_{n_t}(m_{n_t}) = a_{n_{n_t}}$, а для всех остальных случаев

$$\lambda(a) = \lambda(\ldots, \pi_{1_l}(m_{1_l}), \ldots; \ldots, \pi_{r_s}(m_{r_s}), \ldots; \ldots, \pi_{n_t}(m_{n_t}), \ldots) = a.$$

Легко убедиться, что таким образом нами получен упорядоченный автомат **A**. В автомат **A** изоморфно вкладываются все автоматы $\mathbf{M}_r$, если мы поставим в соответствие входному сигналу x_r, состоянию m_r и выходному сигналу y_r автомата $\mathbf{M}_r$ входный сигнал x_r, состояние

$$a = (\ldots, a_{\pi_{1l}} \ldots; \ldots; \ldots, \pi_{r_1}(m_{r_1}), \ldots, \pi_{r_k}(m_{r_k}); \ldots, a_{\pi_{kt}}, \ldots)$$

для которого $\pi_{r_1}(m_{r_1}) \cap \ldots \cap \pi_{r_k}(m_{r_k}) = m_r$ и выходный сигнал y_r автомата **A**. Так как пересечение $\pi_{r_1} \cap \ldots \cap \pi_{r_k}$ совпадает с тривиальным разбиением, то рассматриваемое соответствие — взаимно однозначно. Легко видеть, что оно является и изоморфным.

Анализируя доказательство второй части теоремы 1, можно видеть, что в том случае, когда множество M состоит лишь из одного автомата $\mathbf{M}_1$, теорема 1 получит следующую форму:

Теорема 1'. Если автомат $\mathbf{M}_1$ представляется R-произведением **A** автоматов $\mathbf{A}_i$ ($i=1, \ldots, k$), то каждое множество $P(\mathbf{A}_i)$ индуцирует допустимое разбиение π_{1i} автомата $\mathbf{M}_1$, где состояния из $\mathbf{M}_1$ тогда и только тогда содержатся в одном и том же класса разбиения π_{1i}, если компоненты их образов в данном изоморфизме вложения $\mathbf{M}_1$ в **A**, взятые из любого $\mathbf{A}_j\in P(\mathbf{A}_i)$, совпадают. Если $\mathbf{A}_i \geqslant \mathbf{A}_j$, то разбиение с индуцирующим множеством $P(\mathbf{A}_i)$ больше разбиения, индуцирующегося множеством $P(\mathbf{A}_j)$. Пересечение всех таких разбиений в $\mathbf{M}_1$ является тривиальным.

Обратно, пусть даны допустимые разбиения $\pi_{11}, \ldots, \pi_{1r}$ в автомате $\mathbf{M}_1$, пересечение которых является тривиальным.

Тогда существуют конечное множество автоматов $\{\mathbf{A}_i\}$ $(i=1,\ldots,k)$ и упорядочение R этого множества, такие, что $\mathbf{M}_1$ представляется некоторым R-произведением автоматов $\mathbf{A}_i$, причем каждому разбиению π_{1i} соответствует некоторое $\mathbf{A}_i$ так, что разбиение в $\mathbf{M}_1$, индуцирующееся множеством $P(\mathbf{A}_i)$, совпадает с π_{1i}. $\pi_{1i} \leqq \pi_{1j}$, если для индуцирующих $P(\mathbf{A}_i)$ и $P(\mathbf{A}_j)$ имеет место $\mathbf{A}_i \leqq \mathbf{A}_j$.

Специалным случаем этой теоремы является теорема Хартманиса [2].

Прежде доказательства этого факта мы введем несколько определений. Пусть дано конечное множество автоматов $\{\mathbf{A}_i\}$ $(i=1, \ldots, k)$. Рассмотрим на этом множестве бинарное отношение $\bar{R}: \mathbf{A}_i \bar{R} \mathbf{A}_j$; если произвольный выходный сигнал автомата $\mathbf{A}_i$ является входным сигналом автомата $\mathbf{A}_j$. Транзитивная оболочка этого отношения — частичная упорядоченность R. Если для некоторого автомата $\mathbf{A}_i$ не существует автомат из $\{\mathbf{A}_i\}$, опережающий его, то $\mathbf{A}_i$ называется максимальным. Если автомат $\mathbf{A}_k$ не опережает никакого автомата из $\{\mathbf{A}_i\}$, то $\mathbf{A}_k$ называется минимальным.

Теперь мы сформулируем условия, при которых теорема 1′ перейдет в теорему Хартманиса:

1) В автоматах $\mathbf{M}_1$ и $\mathbf{A}_i$ все входные сигналы, состояния и выходные сигналы — изолированы.

2) Множества X входных сигналов максимальных автоматов совпадают.

3) Множество X_i входных сигналов произвольного автомата $\mathbf{A}_i$ совпадает с множеством $Y_{j1} \times \ldots \times Y_{jl} \times X$, где $Y_{j1}, \ldots, Y_{jl}$ — множества выходных сигналов автоматов $\mathbf{A}_{j1}, \ldots, \mathbf{A}_{jl}$, опережающие автомата $\mathbf{A}_i$,

4) R-произведение автоматов $\mathbf{A}_i$ $(i=1, \ldots, k)$ является автоматом $\mathbf{A}$, множество входных сигналов (соотв. состояний) которого — множество X (соотв. $A_1 \times \ldots \times A_k$). А относительно функции φ обратной связи мы имеем:

$$\varphi(\ldots; a_{i1}, \ldots, a_{ir}, a_i, \ldots, a_k, x) = (x_1, \ldots, x_k),$$

где
$$x_i = \varphi_i(a_{i1}, \ldots, a_{ir}); \, a_{i1} \in \mathbf{A}_{i1}, \ldots, a_{ir} \in \mathbf{A}_{ir},$$

где $\mathbf{A}_{i1}, \ldots, \mathbf{A}_{ir}$ — множество всех автоматов, опережающих автомат $\mathbf{A}_i$. Определение функции φ_i — аналогично определению функции φ_i, использованной при доказательства второй части теоремы 1.

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Universale Algebren mit gegebenen Automorphismengruppen und Unteralgebrenverbänden

Von E. T. SCHMIDT in Budapest

1. In dieser Note wird ein Satz über die Unabhängigkeit der Automorphismengruppen und Unteralgebrenverbänden der universalen Algebren bewiesen.

Eine universale Algebra — oder kurz Algebra — ist ein Paar (A, F), gebildet durch eine Menge A und durch die Gesamtheit F der über A definierten endlichstelligen Operationen. Es ist bekannt (BIRKHOFF [1]), daß jede Gruppe G mit der Automorphismengruppe $G(A, F)$ einer Algebra (A, F) isomorph ist.

Es sei (A, F) eine Algebra und B eine nichtleere Teilmenge von A. Wenn für jedes $\varphi(x_1, x_2, \ldots, x_n) \in F$ aus $x_1, x_2, \ldots, x_n \in B$ stets $\varphi(x_1, x_2, \ldots, x_n) \in B$ folgt, dann wird (B, F) eine Unteralgebra von (A, F) genannt. Es ist bekannt, daß die Gesamtheit der Unteralgebren einer Algebra, in der eine in jeder Unteralgebra enthaltende Unteralgebra existiert, einen kompakt-erzeugten Verband, den sog. Unteralgebrenverband bilden. Nach einem Satz von G. BIRKHOFF und O. FRINK [2] ist jeder kompakt-erzeugte Verband mit dem Unteralgebrenverband $R(A, F)$ einer Algebra (A, F) isomorph.

Wir beweisen den folgenden

Satz. *Es sei G eine beliebige Gruppe und V ein kompakt-erzeugter Verband. Es existiert eine Algebra (A, F) mit $G(A; F) \cong G$ und $R(A, F) \cong V$.*

2. Zum Beweis des Satzes benötigen wir zwei Hilfssätze.

Hilfssatz 1. *Es sei V ein kompakt-erzeugter Verband und H die Menge aller kompakten Elemente von V. Dann ist H ein Halbverband mit Nullelement, und der Idealverband $I(H)$ von H ist mit V isomorph.*

Beweis: Siehe z. B. [3].

Hilfssatz 2. *Zu jeder Gruppe G gibt es eine Algebra (C, F_1) derart, so daß $G(C, F_1) \cong G$ ist und (C, F_1) nur eine von (C, F_1) verschiedene Unteralgebra hat.*

Beweis. Man betrachte die Menge C, die aus G und aus einem neuen Element 0 besteht und definiere auf C die folgenden Operationen:

1. für jedes $a \in G (\subset C)$ je eine Operation $f_a(x)$ durch

$$f_a(x) = \begin{cases} x \cdot a & \text{für } x \in G, \\ 0 & \text{für } x = 0; \end{cases}$$

2. eine binäre Operation $x \cap y$ mit

$$x \cap y = \begin{cases} 0 \text{ für } x \neq y, \\ x \text{ für } x = y \end{cases}$$

(diese Operation ist offensichtlich kommutativ, assoziativ und idempotent).

Es sei $F_1 = \{f_a, \cap\}$. Zuerst zeigen wir, daß $G(C, F_1) \cong G$. Ist α_a $(a \in G)$ eine Abbildung von (C, F_1) in sich, gegeben durch die Regel

$$\alpha_a(x) = \begin{cases} a \cdot x \text{ für } x \neq 0, \\ 0 \text{ für } x = 0, \end{cases}$$

so ist α_a ein Automorphismus: in der Tat, ist $f_a(\alpha_b(x)) = \alpha_b(f_a(x))$ und $\alpha_a(x \cap y) = = \alpha_a(x) \cap \alpha_a(y)$. Umgekehrt, sei α ein beliebiger Automorphismus von (C, F_1). Dann ist 0 ein Fixelement von α (d. h. $\alpha(0) = 0$). Nach der Definition der $\cap$-Operation gilt nämlich $\alpha(0) \cap 0 = 0$ (unabhängig davon, ob $\alpha(0) = 0$ oder $\alpha(0) \neq 0$ ist), woraus sich $\alpha(\alpha(0) \cap 0) = \alpha(\alpha(0)) \cap \alpha(0) = \alpha(0)$ und daraus, im Fall $\alpha(\alpha(0)) \neq \alpha(0)$ auch $\alpha(0) = \alpha(\alpha(0)) \cap \alpha(0) = 0$ ergibt; andererseits, im Fall $\alpha(\alpha(0)) = \alpha(0)$ gilt offenbar auch $\alpha(0) = 0$. Es bezeichne e das Einselement von G und a das Bild von e bei α. Dann ist $\alpha(x) = \alpha(f_x(e)) = f_x(\alpha(e)) = f_x(a) = a \cdot x = \alpha_a(x)$, d. h. jedes Automorphismus stimmt mit einem α_a überein.

Jetzt können wir schon leicht zeigen, daß die Abbildung $a \to \alpha_a$ $(a \in G)$ ein Isomorphismus von G auf $G(C, F_1)$ ist. Nach der entsprechenden Definition gilt nämlich $\alpha_a \alpha_b(x) = ab \cdot x = \alpha_{ab}(x)$, so daß $\alpha_a \alpha_b = \alpha_{ab}$ ist; ferner gilt für $a \neq b$ immer $\alpha_a(e) \neq \alpha_b(e)$.

Wir haben noch zu beweisen, daß die oben definierte Algebra (C, F_1) nur eine von ihr verschiedene Unteralgebra hat. Die einelementige Menge $\{0\}$ ist offensichtlich eine Unteralgebra von (C, F_1). Ist (K, F_1) eine von $\{0\}$ verschiedene Unteralgebra im (C, F_1) so gibt es ein $c \neq 0$ in K. Es sei b ein beliebiges Element aus G. Dann gilt $b = f_{c^{-1}b}(c)$, d. h. $b \in K$. Ist noch $b \neq c$, so folgt auch $0 = b \cap c \in K$. Damit haben wir gezeigt, daß $C = K$ ist. (Hier wurde vorausgesetzt, daß G mindestens zwei verschiedene Elemente enthält; im Fall $G = \{e\}$ darf man $(C, F_1) = \{G, \cdot\}$ wählen.)

3. Jetzt seien G und V gegeben. (C, F_1) sei wie im vorangehenden Paragraphen definiert; H bezeichne den Halbverband der kompakten Elemente von V, und 0 das Nullelement von H.

Besteht V (und folglich auch H) nur aus einem Element, so sei $(A, F) = (G, f_a)$. In diesem Fall die Isomorphie zwischen $G(A, F)$ und G ebenso bewiesen werden, wie im Beweis von Hilfssatz 2. Ferner hat (A, F) keine echte Unteralgebra, da für beliebiges $b \in A$ aus $c \in (A, F)$ auch $b = f_{c^{-1}}(c)$ folgt.

Von hier an können wir voraussetzen, daß V mindestens zwei verschiedene Elemente enthält.

Wir ordnen jedem $h \in H \setminus \{o\}$ eine mit (C, F_1) isomorphe Algebra (C_h, F_1) zu und es bezeichne c_h dasjenige Element in C_h, welches dem Element c von C bei dem Isomorphismus $(C, F_1) \cong (C_h, F_1)$ entspricht. Die Elemente O_h seien identifiziert und — wie in (C, F_1) — einfach durch 0 bezeichnet; übrigens sei aber $C_h \wedge C_k = 0$ für $h \neq k$ vorausgesetzt. Wir definieren eine Algebra (A, F) wie folgt: A sei die

Vereinigungsmenge aller C_h, also $A = \bigvee_{h \in H \setminus \{o\}} C_h$, und die folgenden Operationen seien in Betracht genommen:

1. die ursprünglichen Operationen f_a $(a \in G)$;

2. die ursprüngliche Operation $\cap$, erweitert auf die ganze Menge A durch die Regel: $c_h \cap c'_k = 0$ für $h \neq k$;

3. mit Hilfe der Operation V von H definieren wir eine binäre, assoziative und idempotente Operation auf A wie folgt: $c_h \vee c'_k = c_{h \vee k}$ für $(c' \neq 0)$ und $c_h \vee 0 = c_h$;

4. endlich sei g_h für jedes $h \in H \setminus \{o\}$ durch

$$g_h(c_k) = \begin{cases} c_h & \text{für } k \geqq h \\ 0 & \text{sonst} \end{cases}$$

definiert.

Es sei also $F = \{f_a, \cap, \vee, g_h\}$.

Zuerst zeigen wir, daß $G(A, F) \cong G$ ist. Zu diesem Zweck betrachten wir ein beliebiges Automorphismus α von (A, F). Für $c_h \neq 0$ gilt $\alpha(c_h) \in (C_h, F_1)$: ist nämlich $\alpha(c_h) = c'_k$ so folgt, aus $\alpha(g_h(c_h)) = g_h(\alpha(c_h))$, $c'_k = g_h(c'_k)$ woraus sich entweder $c'_h = c'_k$ $(h = k)$ oder $c'_k = 0$ ergibt. Wäre aber $c'_k = 0$, so wäre $\alpha(c_h) = 0$ und folglich $\alpha(0) \neq 0$; aus $c_h \vee 0 = c_h$ ergibt sich aber $\alpha(c_h) \vee \alpha(0) = \alpha(c_h)$, für jedes c_h und folglich $\alpha(0) = 0$, so daß nur der Fall $c'_h = c'_k$ ist möglich. So ist $\alpha(c_h) \in (C_h, F_1)$ wie wir behauptet haben. Damit haben wir bewiesen, daß α auf jedem (C_h, F_1) einen Automorphismus induziert.

Wir zeigen: Ist $\alpha(c_h) = c'_h$, so gilt $\alpha(c_k) = c'_k$ für jedes $k \in H \setminus \{o\}$. In der Tat gilt $\alpha(c_{h \vee k}) = \alpha(c_h \vee c_k) = \alpha(c_h) \vee \alpha(c_k) = (\alpha(c_h))_{h \vee k} = c'_{h \vee k}$ und folglich $\alpha(c_k) = \alpha(g_k(c_{h \vee k})) = g_k(\alpha(c_{h \vee k})) = g_k(c'_{h \vee k}) = c'_k$.

Wir müssen noch zeigen: ist α_a ein beliebiges Automorphismus von (C, F_1), so läßt sich er auf (A, F) erweitern. Es sei die Abbildung α von (A, F) in sich folgendermaßen definiert:

$\alpha(c_k) = (\alpha(c))_k$ für jedes $k \in M \setminus \{o\}$. Es ist leicht einzusehen, daß dieses α ein Automorphismus von (A, F) und zwar die Erweiterung von α ist. Damit haben wir die Isomorphismus $G \cong G(A, F)$ bewiesen.

Nun beweisen wir, daß $R(A, F) \cong V$ ist.

Es sei K eine von $\{0\}$ verschiedene Unteralgebra von (A, F) und sei $c_h \in K$ für irgendwelches $c \neq 0$ $(h \in H \setminus \{o\})$. Da $K \wedge (C_h, F_1)$ eine Unteralgebra von (C_h, F_1) ist, soll $K \wedge (C_h, F_1) = (C_h, F_1)$ bestehen, und dies bedeutet $c'_h \in K$ für jedes $c' \in (C, F_1)$. Es sei $k \in M \setminus \{o\}$, $k \leqq h$. Aus $c_h \in K$ folgt $c_k = g_k(c_h) \in K$. Ist $c_h, c_k \in K$, so gilt auch $c_{h \vee k} = c_h \vee c_k \in K$. Damit haben wir folgendes bewiesen: aus $c_h, c_k \in K$ folgt $c'_l \in K$ für jedes $c' \in (C, F_1)$ und für jedes l mit $l \leqq h \cup k$. Anders gesagt, bilden das Element O und diejenigen Elemente $h \in M \setminus \{o\}$, für welche $c_h \in K$, ein Ideal von H.

Es sei I ein beliebiges, von Null verschiedenes Ideal von H. Betrachten wir die Teilmenge $K_I = \{0\}\} \vee \{c_h\}_{h \in I, c \in (C, F)}$ von (A, F).

K_I ist offensichtlich eine Unteralgebra von (A, F). Folglich ist $I \to K_I$ eine ein-eindeutige Abbildung von $I(H)$ auf $R(A, F)$ (dem Ideal $(o]$ entspricht die Unteralgebra $\{0\}$) und ist $I(H) \cong R(A, F)$. Nach Hilfssatz 1 ist jedoch $V \cong I(H)$. woraus $V \cong R(A, F)$ folgt. Damit ist unser Satz bewiesen.

4. Es sei (A, F) eine Algebra und $M(A, F)$ die Gesamtheit der Meromorphismen von (A, F) (Ein Meromorphismus ist ein Isomorphismus einer Algebra in sich.) Die Menge $M(A, F)$ ist eine Halbgruppe mit Einselement und ist charakterisiert durch das Erfüllstein der rechtseitige Kürzungsregel (das bedeutet, daß aus $\beta\alpha=\gamma\alpha$ folgt $\beta=\gamma$). Nun stellen wir das folgende

Problem: Es sei V ein kompakt-erzeugter Verband und G eine Halbgruppe mit Einselement, so daß in G die rechtsseitige Kürzungsregel gilt. Gibt es eine Algebra (A, F), so daß $M(A, F)\cong G$ und $R(A, F)\cong V$ ist?

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An application of the theory of regressive functions

By G. FODOR in Szeged

Let E be an arbitrary set of power $\aleph_\alpha$ and suppose that with every element x of E there is associated a non empty set $f(x)$ such that for any $x \in E$ the power of the set $f(x)$ is smaller than a given cardinal number $\aleph_\beta$ which is smaller than $\aleph_\alpha$. We say that the subset Γ of E has the property $T(\mathfrak{q}, \mathfrak{p})$, where $\mathfrak{q}$ and $\mathfrak{p}$ are two cardinal numbers such that $\mathfrak{p} \leqq \mathfrak{q} \leqq \aleph_\alpha$, if $\overline{\overline{\Gamma}} = \mathfrak{q}$ and

$$\overline{\overline{\bigcup_{x, y \in \Gamma} (f(x) \cap f(y))}} < \mathfrak{p}.$$

Consider the following

Proposition. *Under the above conditions E has a subset Γ with the property $T(\aleph_\alpha, \aleph_\alpha)$.*

This proposition was proved in [1] for $\aleph_\alpha$ not the sum of $\aleph_\beta$ or fewer cardinal numbers less than $\aleph_\alpha$, for $\aleph_\alpha$ of the form $\aleph_{\gamma+\omega}$ and — using the generalized continuum hypothesis — in the remaining case too.

We define the sequence $\{\gamma_n\}_{n<\omega}$ as follows:

$$\gamma_1 = \omega_\gamma, \quad \gamma_2 = \omega_{\gamma_1}, \quad \ldots, \quad \gamma_{n+1} = \omega_{\gamma_n}, \ \ldots.$$

We shall prove in this paper the following

Theorem. *If*

a) $\operatorname{cf}(\gamma) > 0$, $\alpha = \gamma_{n+1}$ *and* $\beta < \gamma_n$, *or* b) $\operatorname{cf}(\alpha) > 0$ *and* $\omega_\alpha = \alpha$,

then the above proposition is true.

We shall use the following notations. For any subset Γ of E let

$$\Pi_\Gamma = \bigcup_{\substack{x, y \in \Gamma \\ x \neq y}} (f(x) \cap f(y)).$$

For any cardinal number $\mathfrak{r}$ we denote by $\mathfrak{r}^+$ the cardinal number immediately following $\mathfrak{r}$. The symbols $\overline{\overline{S}}$ and $\bar{\gamma}$ denote the cardinal number of the set S and the ordinal number γ, respectively. For every ordinal number τ, $\aleph_{cf(\tau)}$ denotes the least cardinal number $\mathfrak{n}$ such that $\aleph_\tau$ can be expressed as the sum of $\mathfrak{n}$ cardinal numbers each $< \aleph_\tau$.

By the proof of the theorem we shall use the following

Theorem 1. *If $\aleph_\alpha$ is not the sum of $\aleph_\beta$ or fewer cardinal numbers less than $\aleph_\alpha$, then the above proposition is true.* (See [1], theorem 1.)

Theorem 2. *Let $\aleph_\alpha$ be a singular cardinal number, $\mathfrak{r}_0$ a cardinal number which is smaller than $\aleph_\alpha$ and $\{\aleph_{\nu_\xi}\}_{\xi<\omega_{cf(\alpha)}}$ a sequence of regular cardinal numbers such that $\aleph_{\nu_\sigma}>\aleph_{\nu_\tau}$ $(\sigma>\tau)$, $\max\{\aleph_{cf(\alpha)}, \aleph_\beta, \mathfrak{r}_0\}<\aleph_{\nu_\xi}<\aleph_\alpha$ and $\aleph_\alpha=\sum_{\xi<\omega_{cf(\alpha)}}\aleph_{\nu_\xi}$. If, for every $\xi<\omega_{cf(\alpha)}$, E_ξ is a subset of power $\geqq\aleph_{\nu_\xi}$ of E such that E_ξ has a subset E'_ξ with the property $T(\aleph_{\nu_\xi}, \mathfrak{r}_0)$, then E has a subset with the property $T(\aleph_\alpha, [\aleph_{cf(\alpha)}\cdot\mathfrak{r}_0]^+)$.* (See [1], theorem 4.)

Theorem 3. *Let ω_α be an initial number which is not confinal to ω and M a subset of $W(\omega_\alpha)=\{\eta:\eta<\omega_\alpha\}$. Suppose that to every element $\alpha\in M$ there corresponds an ordinal number $g(\gamma)$ such that $g(\gamma)<\gamma$ for $\gamma>0$ (and $g(0)=0$ for $0\in M$). If $W(\omega_\alpha)-M$ does not contain a closed subset confinal to $W(\omega_\alpha)$ (i. e. M is a stationary subset of $W(\omega_\alpha)$), then there exists an ordinal number $\pi<\omega_\alpha$ and a stationary subset N of M such that $g(\gamma)\leqq\pi$ for every $\gamma\in N$.* (See [2], theorem 2.)

Theorem 4. *Let ω_α be an initial number $>\omega$ and ϱ a regular ordinal number of the second kind with $\varrho<\omega_\alpha$. The set of all ordinal numbers $\lambda<\omega_\alpha$ of the second kind which are confinal to ϱ, is a stationary subset of $W(\omega_\alpha)$.* (See [3], theorem 8.)

Proof of the theorem. We are going to prove a). The proof of b) is quite similar and will be omitted. Let

$$x_0, x_1, \ldots, x_\omega, x_{\omega+1}, \ldots, x_\xi, \ldots \qquad (\xi<\omega_\alpha)$$

be a well-ordering of the type ω_α of the set E. By the hypothesis, $\beta<\gamma_n$. Hence $\beta+1<\gamma_n$ i. e. $\omega_{\beta+1}<\omega_{\gamma_n}=\gamma_{n+1}=\alpha$. Let now $M=\{\mu_\nu\}_{\nu<\alpha}$ be the set of all ordinal numbers of the second kind of $W(\alpha)$ which are confinal to $\omega_{\beta+1}$. By theorem 4 M is a stationary subset of $W(\alpha)$. Put

$$E_\nu=\{x_\eta: \eta<\omega_{\mu_\nu}\}.$$

Obviously $\overline{\overline{E_\nu}}=\aleph_{\mu_\nu}$. Since $\aleph_{\mu_\nu}$ is not the sum of $\aleph_\beta$ or fewer cardinal number less than $\aleph_{\mu_\nu}$ there exists, by theorem 1. a subset Γ_ν of E_ν with the property $T(\aleph_{\mu_\nu}, \aleph_{\mu_\nu})$. Hence, the power of the set Π_{Γ_ν} is smaller than $\aleph_{\mu_\nu}$.

Put $\overline{\overline{\Pi_{\Gamma_\nu}}}=\aleph_{\varrho_\nu}$ and $g(\mu_\nu)=\varrho_\nu$. Thus we have associated with every element μ_ν of M an ordinal number $g(\mu_\nu)$ such that $g(\mu_\nu)<\mu_\nu$ for every $\mu_\nu\in M$. By theorem 3 there exists an ordinal number $\pi<\alpha$ and a stationary subset M' of M such that $g(\mu_\nu)\leqq\pi$ for every $\mu_\nu\in M'$.

Let $\{\mu_{\nu_\eta}\}_{\eta<\omega_{cf(\gamma)}}$ be a subset of the type $\omega_{cf(\gamma)}$ of M' such that $\lim\mu_{\nu_\eta}=\alpha$.

Consider now an increasing sequence $\{\aleph_{\tau_{\nu_\eta}}\}_{\eta<\omega_{cf(\gamma)}}$ of regular cardinal numbers $<\aleph_\alpha$ such that for every $\eta<\omega_{cf(\gamma)}$,

$$\aleph_{\mu_{\nu_\eta}}<\aleph_{\tau_{\nu_\eta}}\leqq\aleph_{\mu_{\nu_{\eta+1}}}.$$

Let Γ'_{ν_η} be a subset of power $\aleph_{\tau_{\nu_\eta}}$ of $\Gamma_{\nu_{\eta+1}}$. It is obvious that $\overline{\overline{\Pi}}_{\Gamma'_{\nu_\eta}} \leqq \bar{\pi} = \mathfrak{r}_0$. Thus by theorem 2 there exists a subset of E with the property $T(\aleph_\alpha, [\aleph_{cf(\alpha)}\mathfrak{r}_0]^+)$.

The theorem is proved.

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Zur Eindeutigkeit der Lösungen der quantenmechanischen Vertauschungsrelationen*)

Von HEINZ GÜNTHER TILLMANN in Mainz (Deutschland)

Quantenmechanische Systeme und quantisierte Wellenfelder können beschrieben werden durch ein System $\mathfrak{S}=\{P_\nu, Q_\nu, \Psi_\lambda, \Psi_\lambda^*\}$[1]) von Operatoren in einem Hilbertschen Raum **H**. Dabei sollen P_ν, Q_ν hermitesche Operatoren sein, welche den „Kommutatorrelationen"

$$P_\mu Q_\nu - Q_\nu P_\mu = -i\delta_{\nu\mu} I, \quad P_\mu P_\nu - P_\nu P_\mu = O, \quad Q_\mu Q_\nu - Q_\nu Q_\mu = O$$

genügen. Ψ_λ und Ψ_λ^* sind zueinander adjungierte Operatoren und sollen den „Antikommutatorrelationen" genügen:

$$\Psi_\lambda \Psi_\varkappa^* + \Psi_\varkappa^* \Psi_\lambda = \delta_{\varkappa\lambda} I, \quad \Psi_\lambda \Psi_\varkappa + \Psi_\varkappa \Psi_\lambda = O.$$

Zu $\mathfrak{S}$ hinzu tritt dann noch der Hamiltonoperator H des Systems, der gewöhnlich als Funktion der Operatoren aus $\mathfrak{S}$ dargestellt wird und die zeitliche Veränderung der Observablen als infinitesimaler Operator beschreibt.

Die physikalisch sinnvollen Aussagen der Theorie, wie Übergangswahrscheinlichkeiten und Erwartungswerte, lassen sich ausdrücken durch innere Produkte $\langle B\varphi, \varphi'\rangle$ oder $\langle B\varphi, \varphi\rangle$, bleiben also bei unitären Transformationen

$$B \to UBU^*, \quad \varphi \to U\varphi$$

invariant.

Es ist deshalb die Frage von Interesse, ob durch die Vertauschungsrelationen (zusammen mit der Form des Hamiltonoperators als Funktion der Operatoren aus $\mathfrak{S}$) die Theorie vollständig festgelegt und die Operatoren aus $\mathfrak{S}$ etwa bis auf eine gemeinsame unitäre Transformation bestimmt sind, bzw. welche zusätzlichen Forderungen man dazu noch stellen muß.

Bekanntlich[2]) ist die Kommutatorrelation $PQ-QP=-iI$ nicht durch Elemente einer Banachalgebra, also nicht durch beschränkte Operatoren im Hilbertraum lösbar. Man muß also zulassen, daß die P_ν, Q_ν unbeschränkte, hermitesche Operatoren sind. Es kann jedoch angenommen werden, daß P_ν, Q_ν abgeschlossene Opera-

* Eine erste Fassung dieser Arbeit entstand 1954—55 während der Tätigkeit des Verfassers am University College Ibadan (Nigeria).

Herrn Prof. B. Sz.-Nagy danke ich sehr für eine eingehende Diskussion während seines Aufenthaltes in Heidelberg im Jahre 1961, welche den Anstoß zur jetzigen, erweiterten Fassung lieferte.

[1]) Vgl. etwa Wentzel, *Einführung in die Quantentheorie der Wellenfelder* (Wien, 1943), oder Bogoljubov—Shirkov, *Introduction to the theory of quantized fields* (New York, 1959).

[2]) Vgl. H. Wielandt [8] (Literaturverzeichnis am Schluß der Arbeit).

toren[3]) sind. Aber solche Operatoren können nicht im ganzen Hilbertraum **H**, sondern nur in dichten Teilmannigfaltigkeiten von **H** definiert sein. Daher erfordert die Kommutatorrelation eine Präzisierung, weil auf beiden Seiten der Gleichung Operatoren mit verschiedenen Definitionsbereichen stehen. Man könnte etwa fordern $PQ\varphi - QP\varphi = -i\varphi$ für $\varphi \in D(PQ) \cap D(QP)$[4]), d. h.

$$(\alpha) \qquad (PQ - QP)^{\sim} = -iI.$$

Hieraus folgt für $\varphi \in D(PQ) \cap D(QP)$, $\varphi' \in D(P) \cap D(Q)$:

$$(\beta) \qquad \langle Q\varphi, P\varphi'\rangle - \langle P\varphi, Q\varphi'\rangle = -i\langle \varphi, \varphi'\rangle.$$

Alle Glieder in (β) sind aber auch für $\varphi \in D(P) \cap D(Q)$ definiert und es erscheint als natürliche Präzisierung Kommutatorrelation zu verlangen, daß (β) auch für alle $\varphi, \varphi' \in D(P) \cap D(Q)$ gültig ist.

In den physikalischen Theorien spielen ferner die Operatoren

$$A = \frac{1}{\sqrt{2}}(Q + iP) \quad \text{und} \quad A' = \frac{1}{\sqrt{2}}(Q - iP) \subset \frac{1}{\sqrt{2}}(Q + iP)^* = A^*$$

eine entscheidende Rolle. Es liegt nur nahe, daß man neben[5]) $A^{**} = A^{\sim} = \{A^{**} | D(P) \cap D(Q)\}^{\sim}$ auch verlangt, daß A^* durch seine Werte auf $D(P) \cap D(Q)$ schon vollständig bestimmt ist, nämlich

$$(\gamma) \qquad A^* = \{A^* | D(P) \cap D(Q)\}^{\sim} = A'^{\sim}.$$

Unter diesen Voraussetzungen können wir zeigen (Theorem 1) daß P und Q im wesentlichen (d. h. bis auf unitäre Äquivalenz und direkte Summenbildung) eindeutig bestimmt sind.

Ein Beispiel zeigt, daß weder auf die Forderung $A^* = A'^{\sim}$ verzichtet noch (β) durch (α) ersetzt werden kann, ohne diese Eindeutigkeit aufzugeben.

Im Falle eines endlichen ober abzählbaren Systems $\mathfrak{S}$ gilt ein ähnlicher Eindeutigkeitssatz (Theorem 3) unter den Voraussetzungen:

a) P_ν, Q_ν erfüllen die Bedingungen (β) und (γ) (für jedes ν);
b) P_ν und Q_ν sind vertauschbar mit P_μ, Q_μ, Ψ_λ, Ψ_λ^* ($\mu \neq \nu$);
c) Ψ_λ, Ψ_λ^* erfüllen die „Antikommutatorrelationen";

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[3]) Ein Operator T heißt abgeschlossen, wenn sein Graph, d. h. die Menge aller Paare $\{\varphi, T\varphi\}$ in $\mathbf{H} \times \mathbf{H}$ abgeschlossen ist. Gleichwertig damit ist die folgende Bedingung: Aus $\varphi_n \to \varphi$, $T\varphi_n \to \tilde{\varphi}$ folgt $\varphi \in D_T$ und $T\varphi = \tilde{\varphi}$. Ist T nicht abgeschlossen, der adjungierte Operator T^* dicht definiert, so ist $T^{\sim} = T^{**}$ eine abgeschlossene Erweiterung von T und zwar diejenige mit kleinstem Definitionsbereich. Insbesondere ist für jeden hermiteschen (nicht notwendig beschränkten) Operator T die Abschliessung $T^{\sim} = T^{**}$ ein abgeschlossener hermitescher Operator.

[4]) Den Definitionsbereich eines Operators T bezeichnen wir mit D_T oder auch mit $D(T)$. J. v. NEUMANN hat in [5] die durch die Unbeschränktheit der P, Q bedingte Problematik der Definitionsbereiche dadurch umgangen, daß er zu den durch die P bzw. Q erzeugten unitären Halbgruppen $U(s) = e^{isP}$, $V(t) = e^{itQ}$ überging und die Vertauschungsrelation (α^*) $U(s)V(t) = e^{ist}V(t)U(s)$ benutzte. Der Übergang von (α) zu (α^*) ist jedoch nur formal möglich. In [4] wurden kürzlich notwendige und hinreichende Zusatzbedingungen für einen strengen Übergang von (α) zu (α^*) angegeben.

[5]) T/D_0 ist die Einschränkung von T auf D_0, d. h. auf $D_0 \cap D(T)$.

d) es existiert ein „Vakuumzustand" φ_0;
e) das System $\mathfrak{S}$ ist irreduzibel[6]) in $\mathbf{H}$.

Auf die beiden letzten Forderungen kann verzichtet werden, falls $\mathfrak{S}$ nur aus endlich vielen Paaren P_ν, Q_ν und Ψ_λ, Ψ_λ^* besteht.

Im Falle eines Freiheitsgrades sind die früheren Resultate von F. Rellich [3] und J. Dixmier [4] in unseren Ergebnissen enthalten (vgl. 7). Im Falle endlich vieler Freiheitsgrade erscheint unser Resultat besonders befriedigend, da wir für die Operatoren mit verschiedenen Indizes außer den Vertauschbarkeitsbedingungen keine weiteren Forderungen zu stellen brauchen, wie es etwa noch bei Dixmier [4] nötig war.

1. Präzisierung der Kommutatorrelation $PQ-QP=-iI$

Da diese Relation nicht durch beschränkte Operatoren in einem Hilbertraum $\mathbf{H}$ erfüllt werden kann, fordern wir

1: P, Q sind abgeschlossene, hermitesche Operatoren in einem Hilbertraum $\mathbf{H}$. $D_P \cap D_Q$ ist dicht in $\mathbf{H}$.

Aus $PQ\varphi - QP\varphi = -i\varphi$ folgt für alle $\varphi' \in D_P \cap D_Q$:

$$\langle Q\varphi, P\varphi'\rangle - \langle P\varphi, Q\varphi'\rangle = -i\langle \varphi, \varphi'\rangle.$$

Umgekehrt folgt hieraus wieder $PQ\varphi - QP\varphi = -i\varphi$, falls nur $PQ\varphi$ und $QP\varphi$ definiert sind. Alle hier auftretenden inneren Produkte sind aber für beliebige $\varphi, \varphi' \in D_P \cap D_Q$ definiert. Als natürliche Präzisierung der Kommutatorrelation erscheint deshalb die Vertauschungsrelation

2: $\langle Q\varphi, P\varphi'\rangle - \langle P\varphi, Q\varphi'\rangle = -i\langle \varphi, \varphi'\rangle$ für alle $\varphi, \varphi' \in D_P \cap D_Q$.

Ist $D_0 := D_{PQ} \cap D_{QP}$ noch dicht in $\mathbf{H}$, so folgt aus **1** und **2**:

2′: $(PQ-QP)^\sim = -iI$.

Wir setzen noch

$$A = \frac{1}{\sqrt{2}}(Q+iP), \quad A' = \frac{1}{\sqrt{2}}(Q-iP) \tag{1}$$

und mit $Q_0 = Q|D_0$, $P_0 = P|D_0$:

$$A_0 = \frac{1}{\sqrt{2}}(Q_0+iP_0)^\sim, \quad A_0' = \frac{1}{\sqrt{2}}(Q_0-iP_0)^\sim. \tag{2}$$

Satz 1. *P, Q seien Operatoren, die den Bedingungen* **1, 2** *genügen. Dann gilt: A und A' sind abgeschlossene Operatoren und $A' \subset A^*$;*

$$A^*A + \frac{1}{2}I = A'^*A' - \frac{1}{2}I \supset \frac{1}{2}(Q^2+P^2); \tag{3}$$

$$AA^* - \frac{1}{2}I \supset AA_0' - \frac{1}{2}I \supset \frac{1}{2}(QQ_0+PP_0). \tag{4}$$

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[6]) $\mathfrak{S}$ heißt irreduzibel, wenn es keinen echten Teilraum $\mathbf{H}_0 \subset \mathbf{H}$ gibt, der alle Operatoren aus $\mathfrak{S}$ reduziert. Dies ist äquivalent damit, daß es keine Projektion gibt, die mit allen Operatoren aus $\mathfrak{S}$ vertauschbar ist.

Beweis. Für alle $\varphi, \varphi' \in D_A = D_{A'} = D_P \cap D_Q$ gilt:

$$(5)\qquad \langle A\varphi, A\varphi'\rangle = \frac{1}{2}\{\langle Q\varphi, Q\varphi'\rangle + \langle P\varphi, P\varphi'\rangle - i(\langle Q\varphi, P\varphi'\rangle - \langle P\varphi, Q\varphi'\rangle)\}$$

$$= \frac{1}{2}\{\langle Q\varphi, Q\varphi'\rangle + \langle Q\varphi, P\varphi'\rangle - \langle \varphi, \varphi'\rangle\} \quad \text{und analog}$$

$$(6)\qquad \langle A'\varphi, A'\varphi'\rangle = \frac{1}{2}\{\langle Q\varphi, Q\varphi'\rangle + \langle P\varphi, P\varphi'\rangle + \langle \varphi, \varphi'\rangle\}, \quad \text{insbesondere}$$

$$(7)\qquad \langle A'\varphi, A'\varphi'\rangle = \langle A\varphi, A\varphi'\rangle + \langle \varphi, \varphi'\rangle,$$

$$(8)\qquad |A\varphi|^2 = \frac{1}{2}\{|Q\varphi|^2 + |P\varphi|^2 - |\varphi|^2\},$$

$$(9)\qquad |A'\varphi|^2 = \frac{1}{2}\{|Q\varphi|^2 + |P\varphi|^2 + |\varphi|^2\}.$$

Da P und Q abgeschlossen sind, folgt aus (8) und (9) die Abgeschlossenheit von A und von A'. Da P und Q hermitesch sind, gilt

$$A^* = \frac{1}{\sqrt{2}}(Q+iP)^* \supset \frac{1}{\sqrt{2}}(Q^* - iP^*) \supset \frac{1}{\sqrt{2}}(Q - iP) = A'.$$

Aus (7) folgt, daß A'^*A' und A^*A den gleichen Definitionsbereich haben und gilt:

$$A^*A + \frac{1}{2}I = A'^*A' - \frac{1}{2}I \supset \frac{1}{2}(Q^2 + P^2).$$

Da im Definitionsbereich von $QQ_0 + PP_0$ gerade alle der Operatoren P^2, PQ, QP, Q^2 definiert sind, folgt unmittelbar aus der Definition von A und A':

$$AA^* - \frac{1}{2}I \supset AA_0' - \frac{1}{2}I \supset \frac{1}{2}(QQ_0 + PP_0)$$

und damit (4). Wir stellen nun noch die zusätzliche Forderung:

3: $(Q+iP)^* = \{(Q+iP)^* | D_P \cap D_Q\}^\sim$ (d. h. $= (Q - iP)^\sim$).

Dies ist unter Voraussetzung von Satz 1 gleichwertig mit $A' = A^*$, und wir erhalten:[7])

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[7]) Aus **1** und **2'** folgt bereits

$$A_0'^* A_0' - \tfrac{1}{2}I = A_0^* A_0 - \tfrac{1}{2}I \supset \tfrac{1}{2}(QQ_0 + PP_0) \subset AA_0' - \tfrac{1}{2}I \subset AA^* - \tfrac{1}{2}I.$$

Satz 2 gilt also auch unter den Voraussetzungen **1** und **2'** zusammen mit (3*) $A^* = (A^*/D_{PQ} \cap \cap D_{QP})^\sim = A_0'$, oder mit (3**) $QQ_0 + PP_0$ ist wesentlich selbstadjungiert. Damit sind auch die Resultate von Rellich [6] und Dixmier [1] in unserem Theorem 1 enthalten.

Es genügt jedoch nicht, **1, 2'** und **3** vorauszusetzen, wie das folgende Beispiel zeigt:

$$\mathbf{H} = L^2(0,1), \quad Q\varphi(x) = x \cdot \varphi(x), \quad D_Q = \mathbf{H}, \quad P\varphi(x) = -i\frac{d}{dx}\varphi(x),$$

$$D_P = \{\varphi : \varphi, \varphi' \in L^2, \ \varphi(0) = \varphi(1)\}.$$

Dann sind P und Q selbstadjungiert und es gilt **1, 2', 3**, aber $D(A^*A) \neq D(AA^*)$.
Zugleich folgt, daß **3** nicht aus **1** und **2** abgeleitet werden kann, denn schränkt man D_P durch die zusätzliche Bedingung $\varphi(0) = \varphi(1) = 0$ ein, so gilt **1** und **2**, aber nicht **3**.

Satz 2. *Erfüllen P, Q die Forderungen* **1**, **2**, **3**, *so gilt:*

$$AA^* = A^*A+I.$$

Satz 3. *Aus*

$$AA^* = A^*A+I \quad (A \text{ abgeschlossen}), \tag{10}$$

folgt: $B=AA^*$ *hat ein rein diskretes Spektrum, bestehend aus den Eigenwerten* $\lambda_n = n+1$ $(n=0, 1, 2, \ldots)$. *Diese haben alle die gleiche Vielfachheit.*

Beweis. Es sei $B = AA^* = A^*A+I$. B ist selbsadjungiert und hat sein Spektrum in $[1, \infty]$. Sei λ_0 aus dem Spektrum von B und $B=\int \lambda dE_\lambda$ die kanonische Spektralzerlegung von B. Dann ist $\{E_\lambda\}$ in der Umgebung von λ_0 nicht konstant, es gibt also normierte Vektoren φ_n, so daß $\varphi_n = \left(E_{\lambda_0+\frac{1}{n}} - E_{\lambda_0-\frac{1}{n}}\right)\varphi_n$, $B\varphi_n = \int\limits_{\lambda_0-\frac{1}{n}}^{\lambda_0+\frac{1}{n}} \lambda dE_\lambda \varphi_n$.

φ_n liegt dann im Definitionsbereich von B^2, insbesondere existiert

$$BA\varphi_n = AA^*A\varphi_n = A(B-I)\varphi_n = (\lambda_0-1)A\varphi_n + A\varphi_n' \tag{11}$$

mit
$$\varphi_n' = \int\limits_{|\lambda-\lambda_0|\leqq\frac{1}{n}} (\lambda-\lambda_0)dE_\lambda\varphi_n, \quad \text{also} \quad |\varphi_n'| \leqq \frac{1}{n}|\varphi_n| = \frac{1}{n},$$

$$|A\varphi_n'|^2 = \langle A^*A\varphi_n', \varphi_n'\rangle = \langle (B-I)\varphi_n', \varphi_n'\rangle \leqq \left(\lambda_0 + \frac{1}{n} - 1\right)|\varphi_n'|^2 \leqq \lambda_0 \cdot \frac{1}{n^2}. \tag{12}$$

Aus (11), (12) folgt also

$$|BA\varphi_n - (\lambda_0-1)A\varphi_n| \to 0, \quad |\varphi_n| = 1. \tag{13}$$

Hat nun die Folge $|A\varphi_n|$ eine positive untere Schranke, so gehört auch $\lambda_0 - 1$ zum Spektrum von B. Existiert eine solche Schranke nicht, so gibt es eine Teilfolge $\{\varphi_{n_k}\}$ für die $|A\varphi_{n_k}|$ gegen Null strebt. Dann folgt aber aus

$$A\varphi_n|^2 = \langle A^*A\varphi_n, \varphi_n\rangle = \langle (B-I)\varphi_n, \varphi_n\rangle \geqq (\lambda_0 - \frac{1}{n} - 1)|\varphi_n|^2 = \lambda_0 - 1 - \frac{1}{n},$$

daß $\lambda_0 = 1$ sein muß.

Alle Spektralwerte sind also natürliche Zahlen und mit $\lambda_0 = k$ sind auch $1, 2, \ldots$ $\ldots, k-1$ Werte des Spektrums von B. Es sei nun $\mathbf{H}_n$ der Eigenraum von B zum Eigenwert $\lambda_n = n+1$. Dann bildet A den Eigenraum $\mathbf{H}_n$ in $\mathbf{H}_{n-1}$ ab und für $n \geqq 1$ ist diese Abbildung eineindeutig. A^* bildet $\mathbf{H}_{n-1}$ eineindeutig in $\mathbf{H}_n$ ab und aus $AA^*\varphi = (n+1)\varphi$, $A^*A\varphi = n\varphi$ für jedes $\varphi \in \mathbf{H}_n$ folgt, daß die Abbildungen $A: \mathbf{H}_n \to$ $\to \mathbf{H}_{n-1}$, $A^*: \mathbf{H}_n \to \mathbf{H}_{n+1}$ Abbildungen „auf" sind. Also haben alle Räume $\mathbf{H}_n$ die gleiche Dimension, die wir mit m bezeichnen.

2. Eindeutigkeit der Operatoren A, A^*, P, Q

Wir setzen nun voraus, daß die Operatoren P, Q so beschaffen sind, daß die Operatoren A und $A' = A^*$ aus (1) der Relation (10)

$$AA^* = A^*A + I$$

genügen. Wir werden zeigen, daß dann der Hilbertraum $\mathbf{H}$ in eine direkte orthogonale Summe $\mathbf{H} = \oplus \mathbf{H}^{(\alpha)}$ von m Teilräumen $\mathbf{H}^{(\alpha)}$ zerlegt werden kann, derart, daß $\mathbf{H}^{(\alpha)}$ die Operatoren P, Q, A, A^* reduziert. Die Einschränkungen von P, Q auf $\mathbf{H}^{(\alpha)}$ sind zu den Heisenberg—Schrödingerschen Operatoren unitär äquivalent.

Zunächst verschaffen wir uns eine geeignete Orthonormalbasis von $\mathbf{H}$.

Satz 4. *Es sei $\mathbf{H}_0$ der Eigenraum von AA^* zum niedrigsten Eigenwert $\lambda_0 = 1$ und $m = \dim \mathbf{H}_0$. $\{\varphi_{\alpha,0}\}$ sei eine Orthonormalbasis in $\mathbf{H}_0$. Dann existiert*

$$\varphi_{\alpha,k} = \frac{1}{\sqrt{k!}} (A^*)^k \varphi_{\alpha,0} \qquad (k = 1, 2, 3, \ldots), \tag{14}$$

$\varphi_{\alpha,k}$ ist Eigenvektor von AA^* zum Eigenwert $\lambda_k = k+1$ und $\{\varphi_{\alpha,k}\}$ ist eine Orthonormalbasis von $\mathbf{H}$.

Beweis. Daß $\varphi_{\alpha,k}$ existiert und Eigenvektor zum Eigenwert $\lambda_k = k+1$ von AA^* ist, wurde bereits im Beweis von Satz 3 gezeigt. Aus $A^*\colon \mathbf{H}_n \xrightarrow{\text{auf}} \mathbf{H}_{n+1}$ folgt weiter, daß $\{\varphi_{\alpha,k}\}$ für festes k in $\mathbf{H}_k$ total ist. Es bleibt die Orthonormalität zu zeigen:

$$\langle \varphi_{\alpha,k}, \varphi_{\beta,k} \rangle = \frac{1}{k!} \langle (A^*)^k \varphi_{\alpha,0}, (A^*)^k \varphi_{\beta,0} \rangle = \frac{1}{(k-1)!} \left\langle \frac{AA^*}{k} (A^*)^{k-1} \varphi_{\alpha,0}, (A^*)^{k-1} \varphi_{\beta,0} \right\rangle$$

$$= \frac{1}{(k-1)!} \langle (A^*)^{k-1} \varphi_{\alpha,0}, (A^*)^{k-1} \varphi_{\beta,0} \rangle = \langle \varphi_{\alpha,k-1}, \varphi_{\beta,k-1} \rangle = \langle \varphi_{\alpha,0}, \varphi_{\beta,0} \rangle$$

$$= \delta_{\alpha,\beta}.$$

Für $k \neq j$ sind aber $\varphi_{\alpha,k}$ und $\varphi_{\beta,j}$ als Eigenvektoren zu verschiedenen Eigenwerten von AA^* auch orthogonal.

Da die Eigenräume $\mathbf{H}_k$ den Raum $\mathbf{H}$ erzeugen, ist $\{\varphi_{\alpha,k}\}$ auch vollständig in $\mathbf{H}$.

Satz 5. *A und A^* sind vollständig charakterisiert durch die Relationen*

$$A^* \varphi_{\alpha,k} = \sqrt{k+1}\, \varphi_{\alpha,k+1}, \quad A\varphi_{\alpha,k} = \sqrt{k}\, \varphi_{\alpha,k-1} \qquad (A\varphi_{\alpha,0} = 0). \tag{15}$$

Der Definitionsbereich von A und A^* ist die Menge

$$D = \{\varphi;\ \varphi = \sum c_{\alpha,k} \varphi_{\alpha,k};\ \sum_k (k+1) \sum_\alpha |c_{\alpha,k}|^2 < \infty\} = D_A = D_{A^*}. \tag{16}$$

Beweis. Aus der Definition der $\varphi_{\alpha,k}$ folgen unmittelbar die Relationen (15). Da A und A^* abgeschlossen sind, ist D sicher in D_A und D_{A^*} enthalten. Es sei nun

$$\varphi = \sum b_{\alpha,k} \varphi_{\alpha,k} \in D_A, \quad A\varphi = \sum \beta_{\alpha,k} \varphi_{\alpha,k},$$

$$\varphi' = \sum c_{\alpha,k} \varphi_{\alpha,k} \in D_{A^*}, \quad A^*\varphi = \sum \gamma_{\alpha,k} \varphi_{\alpha,k}.$$

Dann ist

$$\beta_{\alpha,k}=\langle A\varphi,\ \varphi_{\alpha,k}\rangle=\langle\varphi,\ A^*\varphi_{\alpha,k}\rangle=\sqrt{k+1}\,b_{\alpha,k+1},$$

$$\gamma_{\alpha,k}=\langle A^*\varphi',\ \varphi_{\alpha,k}\rangle=\langle\varphi',\ A\varphi_{\alpha,k}\rangle=\sqrt{k}\,c_{\alpha,k-1}.$$

Es gilt also

$$\sum_{\alpha}\sum_{k=1}^{\infty}|\beta_{\alpha,k-1}|^2=\sum_{k=1}^{\infty}k\sum_{\alpha}|b_{\alpha,k}|^2<\infty,$$

$$\sum_{\alpha}\sum_{k=0}^{\infty}|\gamma_{\alpha,k+1}|^2=\sum_{k=1}^{\infty}(k+1)\sum_{\alpha}|c_{\alpha,k}|^2<\infty.$$

Da aber $\sum_{\alpha}\sum_{k=0}^{\infty}|b_{\alpha,k}|^2<\infty$ ist, ergeben diese beiden Relationen, daß sowohl φ als auch φ' zu D gehören, also $D=D_A=D_{A^*}$ ist.

Bemerkung. Aus den Relationen (15) und (16) folgt mit $Q\supset\frac{1}{\sqrt{2}}(A^*+A)$, $P\supset\frac{i}{\sqrt{2}}(A^*-A)$ unmittelbar, daß der Operator P^2+Q^2 wesentlich selbstadjungiert ist, sowohl in seinem natürlichen Definitionsbereich $D_{P^2}\cap D_{Q^2}$ als auch in der linearen Hülle L der $\varphi_{\alpha,k}$, welche bezüglich P, Q, A und A^* invariant ist.

Theorem 1. *Es seien P, Q abgeschlossene, hermitesche Operatoren, es gelte* **2** *und* **3**, *also auch $AA^*=A^*A+I$. Dann gilt:*

$$Q=\frac{1}{\sqrt{2}}(A^*+A)^{\sim},\quad P=\frac{i}{\sqrt{2}}(A^*-A)^{\sim};\tag{17}$$

P und Q sind selbstadjungiert. Der Hilbertraum **H** *kann zerlegt werden in die direkte Summe vom m paarweise orthogonalen Teilräumen* $\mathbf{H}^{(\alpha)}$, *welche P, Q, A, A^* reduzieren. Die Einschränkungen $P^{(\alpha)}, Q^{(\alpha)}$ von P, Q auf $\mathbf{H}^{(\alpha)}$ sind in $\mathbf{H}^{(\alpha)}$ irreduzibel. Diese irreduziblen Bestandteile sind bis auf unitäre Äquivalenz eindeutig bestimmt.*

Beweis. Auf $D=D_A=D_{A^*}$ (vgl. (16)) gilt wegen (1)

$$Q\varphi=\frac{1}{\sqrt{2}}(A^*+A)\varphi,\quad P\varphi=\frac{i}{\sqrt{2}}(A^*-A)\varphi,$$

also auch

$$Q\supset\frac{1}{\sqrt{2}}(A^*+A)^{\sim},\quad P\supset\frac{i}{\sqrt{2}}(A^*-A)^{\sim}.$$

In **H** ist durch

$$J\left(\sum c_{\alpha,k}\varphi_{\alpha,k}\right)=\sum\overline{c_{\alpha,k}}\varphi_{\alpha,k}$$

eine Konjugation J definiert und $(A^*+A)^{\sim}$ ist bezüglich dieser Konjugation reell besitzt also eine selbstadjungierte Fortsetzung, hat also gleiche Defektzahlen. Wir zeigen, daß die Cayleysche Transformierte von $(A^*+A)^{\sim}$,

$$V=\{(A^*+A)^{\sim}-iI\}\{(A^*+A)^{\sim}+iI\}^{-1}$$

den ganzen Raum **H** als Definitionsbereich hat. Es sei $\varphi_0\perp D_V$. Dann gilt insbesondere:

$\langle \varphi_0, (A^* + A + iI)\varphi_{\alpha,k}\rangle = 0$ und wenn $\varphi_0 = \sum d_{\alpha,k} i^k \varphi_{\alpha,k}$ gilt, so heißt dies:

$$0 = \langle \sum d_{\beta,j} i^j \varphi_{\beta,j}, \sqrt{k}\varphi_{\alpha,k-1} + i\varphi_{\alpha,k} + \sqrt{k+1}\,\varphi_{\alpha,k+1}\rangle,$$

also $$0 = \sqrt{k}\, d_{\alpha,k-1} i^{k-1} + d_{\alpha,k} i^{k-1} + \sqrt{k+1}\, d_{\alpha,k+1} i^{k+1}$$ oder

$$d_{\alpha,k+1} = \frac{1}{\sqrt{k+1}}(d_{\alpha,k} + \sqrt{k}\, d_{\alpha,k-1}). \tag{18}$$

Es sei nun irgendein $d_{\alpha_0,0} \neq 0$. Wir können dann $d_{\alpha_0,0} = 1$ annehmen. Aus (18) folgt dann $d_{\alpha_0,1} = 1$ und durch Induktion: $d_{\alpha_0,k+1} \geqq \frac{1+\sqrt{k}}{\sqrt{k+1}} \geqq 1$, was der Konvergenz von $\sum |d_{\alpha,k}|^2$ widerspricht. Es müssen also alle $d_{\alpha_0,0}$ und nach (18) auch alle $d_{\alpha,k}$ verschwinden. V ist also im ganzen Raum $\mathbf{H}$ definiert und $(A + A^*)^{\sim}$ ist selbstadjungiert. Da aber ein selbstadjungierter Operator keine echte hermitesche Fortsetzung haben kann, folgt

$$Q = \frac{1}{\sqrt{2}}(A^* + A)^{\sim} = Q^*.$$

Setzt man $\psi_{\alpha,k} = i^k \varphi_{\alpha,k}$ so gilt:

$$(A^* + A)\varphi_{\alpha,k} = \sqrt{k+1}\,\varphi_{\alpha,k+1} + \sqrt{k}\,\varphi_{\alpha,k-1} = \sqrt{2}\, Q\varphi_{\alpha,k},$$

$$i(A^* - A)\psi_{\alpha,k} = \sqrt{k+1}\,\psi_{\alpha,k+1} + \sqrt{k}\,\psi_{\alpha,k-1} = \sqrt{2}\, P\psi_{\alpha,k}.$$

Die Operatoren $(A^* + A)^{\sim}$ und $i(A^* - A)^{\sim}$ sind also unitär äquivalent, d. h. $\frac{i}{\sqrt{2}}(A^* - A)^{\sim}$ ist auch selbstadjungiert, stimmt also mit P überein.

Bezeichnen wir mit $\mathbf{H}^{(\alpha)}$ den durch alle $\varphi_{\alpha,k}$ $(k = 0, 1, 2, \ldots)$ aufgespannten Teilraum von $\mathbf{H}$, so werden offenbar P, Q, A und A^* durch $\mathbf{H}^{(\alpha)}$ reduziert. Die Einschränkungen $P^{(\alpha)}, Q^{(\alpha)}$ von P, Q auf $\mathbf{H}^{(\alpha)}$ werden aber bezüglich der Basis $\{\varphi_{\alpha,k}\}$ gerade durch die Heisenbergschen Matrizen charakterisiert, sind also insbesondere bis auf unitäre Äquivalenz eindeutig festgelegt. Die Irreduzibilität von $\{P^{(\alpha)}, Q^{(\alpha)}\}$ ist leicht einzusehen. Ist $\mathbf{M} \subset \mathbf{H}^{(\alpha)}$ bei P, Q invariant, so auch bei A, A^* und AA^*. Also ist die Projektion auf $\mathbf{M}$, E_M, vertauschbar mit den Projektionen E_n auf die Eigenräume $\mathbf{H}_n$ von AA^*. $\mathbf{H}_n \cap \mathbf{H}^{(\alpha)} = \{\lambda\varphi_{\alpha,n}\}$ ist aber eindimensional. Ist $\mathbf{M} \neq 0$, so muß $\mathbf{H}_n \cap \mathbf{M} \neq \{0\}$ sein für ein n, d. h. $\mathbf{M}$ enthält $\varphi_{\alpha,n}$, wegen Invarianz bei A, A^* aber dann auch alle $\varphi_{\alpha,k}$, also ganz $\mathbf{H}^{(\alpha)}$.

3. Lösungen der Antikommutatorrelationen

Die Felder von Fermi—Dirac-Teilchen werden beschrieben durch Operatoren Ψ, Ψ^*, welche den „Antikommutatorrelationen” genügen:

$$\Psi\Psi^* + \Psi^*\Psi = I, \quad \Psi\Psi = \Psi^*\Psi^* = 0. \tag{19}$$

Dabei soll Ψ ein abgeschlossener Operator, Ψ^* sein Adjungierter sein. Es sei nun

$$F^0 = \Psi\Psi^*, \quad F^1 = \Psi^*\Psi = I - F^0. \tag{20}$$

Satz 6. *Wenn Ψ, Ψ^* den Relationen* (19) *genügen, so gilt: F^0 und $F^1 = I - F^0$ (aus* (20)*) sind orthogonale Projektionen auf zueinander orthogonale, komplementäre Teilräume $\mathbf{H}^0, \mathbf{H}^1$ des Hilbertraums $\mathbf{H}$; Ψ (bzw. Ψ^*) ist eine partiell isometrische Abbildung mit dem Anfangsbereich $\mathbf{H}^1$ (bzw. $\mathbf{H}^0$) und dem Endbereich $\mathbf{H}^0$ (bzw. $\mathbf{H}^1$).*

Zu jeder Orthonormalbasis $\{\varphi_{\nu,0}\}$ von $\mathbf{H}^0$ gibt es eine Orthonormalbasis $\{\varphi_{\nu,1}\}$ von $\mathbf{H}^1$, so daß Ψ, Ψ^* dargestellt werden durch die Relationen

$$\Psi^*\varphi_{\nu,0} = \varphi_{\nu,1}, \quad \Psi^*\varphi_{\nu,1} = 0, \tag{21}$$

$$\Psi\varphi_{\nu,0} = 0, \quad \Psi\varphi_{\nu,1} = \varphi_{\nu,0}. \tag{22}$$

Die zweidimensionalen Räume $\mathbf{H}^{(\nu)}$, die durch $\varphi_{\nu,0}$ und $\varphi_{\nu,1}$ aufgespannt werden, reduzieren Ψ und Ψ^*. Ψ und Ψ^* sind also unitär äquivalent zu einer direkten Summe von Matrizen

$$\Psi^{(\nu)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Psi^{(\nu)*} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Beweis. Aus $\Psi\Psi^* = I - \Psi^*\Psi$ folgt $0 \leqq F^j \leqq I$. Also sind F^j und damit Ψ, Ψ^* beschränkte, also auch überall definierte Operatoren.

Aus $\Psi^2 = 0 = (\Psi^*)^2$ folgt $F^1F^0 = F^0F^1 = 0$ und

$$(F^0)^2 = F^0(I - F^1) = F^0; \quad (F^1)^2 = F^1(I - F^0) = F^1.$$

F^j sind also selbstadjungiert und idempotent, also orthogonale Projektionen auf abgeschlossene Teilräume $\mathbf{H}^0$ bzw. $\mathbf{H}^1$. Aus $F^0F^1 = F^1F^0 = 0$ folgt, daß $\mathbf{H}^0 \perp \mathbf{H}^1$, und $F^0 + F^1 = I$ ergibt $\mathbf{H} = \mathbf{H}^0 + \mathbf{H}^1$, also $\mathbf{H}^1 = (\mathbf{H}^0)^\perp$. Da $\Psi^*\Psi$ eine Projektion ist, ist Ψ partiell isometrisch, und zwar bildet es $\mathbf{H}^1$ isometrisch auf $\mathbf{H}^0$ ab:

$$|\Psi\varphi_1|^2 = \langle \Psi^*\Psi\varphi_1, \varphi_1 \rangle = |\varphi_1|^2 \quad \text{für} \quad \varphi_1 \in H^1,$$

$$\Psi\varphi_1 \in H^0, \quad \text{da} \quad F^1\Psi\varphi_1 = \Psi^*(\Psi)^2\varphi_1 = 0 \quad \text{ist.}$$

Analog sieht man, daß Ψ^* den Raum $\mathbf{H}^0$ isometrisch in $\mathbf{H}^1$ abbildet und $\mathbf{H}^1$ annuliert. Da aber F^0 und F^1 Abbildungen auf $\mathbf{H}^0$ bzw. $\mathbf{H}^1$ sind, folgt, daß auch $\Psi\mathbf{H}^1 = \mathbf{H}^0$ und $\Psi^*\mathbf{H}^0 = \mathbf{H}^1$ ist.

Definieren wir zu einer Orthonormalbasis $\{\varphi_{\nu,0}\}$ von $\mathbf{H}^0$ $\varphi_{\nu,1} = \Psi^*\varphi_{\nu,0}$, so sind die Bedingungen (21), (22) erfüllt und $\{\varphi_{\nu,1}\}$ ist eine Orthonormalbasis von $\mathbf{H}^1$.

Die letzte Aussage von Satz 6 entnimmt man nun unmittelbar aus (21), (22).

4. Darstellung der Operatoren von Boson- und Fermion-Feldern

Es sei $\mathfrak{S} = \{P_\mu, Q_\mu, \Psi_\lambda, \Psi_\lambda^*\}_{\mu \in M, \lambda \in N}$ ein höchstens abzählbares System von linearen Operatoren im Hilbertschen Raum $\mathbf{H}$. Dabei sollen die P_μ, Q_μ den „Kommutatorrelationen", Ψ_λ, Ψ_λ^* den „Antikommutatorrelationen" genügen und alle P, Q sollen mit allen Ψ, Ψ^* vertauschbar sein.

Genauer verlangen wir folgendes:

I. Ψ_λ, Ψ_λ^* *sind zueinander adjungiert und für alle* $\lambda, \varkappa \in N$ *gilt:*

$$\Psi_\lambda \Psi_\varkappa^* + \Psi_\varkappa^* \Psi_\lambda = \delta_{\varkappa\lambda} I, \quad \Psi_\lambda \Psi_\varkappa + \Psi_\varkappa \Psi_\lambda = O;$$

II. P_μ, Q_μ *ist für jedes* μ *ein „kanonisches Paar", d. h. es gelten die Bedingungen* **1**, **2**, **3** *aus Nr.* 1;

III. P_μ, Q_μ *sind vertauschbar mit* P_ν, Q_ν, Ψ_λ, Ψ_λ^* *für alle* $\lambda \in N$ *und alle* $\mu \neq \nu \in M$, *d. h. die Spaktralschar von* P_μ *(bzw.* Q_μ*) ist mit der Spektralschar von* P_ν, *von* Q_ν *und mit den Operatoren* Ψ_λ, Ψ_λ^* *(die ja beschränkt sind) vertauschbar im üblichen Sinne.*

Satz 7. *Aus* I *folgt:*

$$F_\lambda^0 = \Psi_\lambda \Psi_\lambda^* = I - \Psi_\lambda^* \Psi_\lambda = I - F_\lambda^1$$

sind orthogonale Projektionen. Die Elemente der Familie $\{F_\lambda^0, F_\lambda^1\}_{\lambda \in N}$ *sind paarweise miteinander vertauschbar.* Ψ_λ *und* Ψ_λ^* *sind partiell isometrische Operatoren, die mit allen* $F_\varkappa^0$, $F_\varkappa^1$, $\varkappa \neq \lambda$, *vertauschbar sind.*

Beweis. Nach Nr. 3 sind die F_λ^j Projektionen, die Ψ_λ und Ψ_λ^* partiell isometrische Operatoren. Aus den Relationen in I für $\lambda \neq \varkappa$ folgt aber: $F_\lambda^j F_\varkappa^i = F_\varkappa^i F_\lambda^j$, $\Psi_\lambda F_\varkappa^j = F_\varkappa^j \Psi_\lambda$ und $\Psi_\lambda^* F_\varkappa^j = F_\varkappa^j \Psi_\lambda^*$. Aus II folgt nach Theorem 1, daß alle P_μ, Q_μ selbstadjungierte Operatoren in **H** sind. Daher hat die Forderung III der Vertauschbarkeit von P_μ mit Q_ν, P_ν, Ψ_λ, Ψ_λ^* einen wohlbestimmten Sinn.

Ist E_0 irgendeine Projektion aus der Spektralschar von P_μ (bzw. von Q_μ), so bedeutet III, daß der Raum $\mathbf{H}_0 = E_0(\mathbf{H})$ die Operatoren P_ν, Q_ν, Ψ_λ, Ψ_λ^* reduziert. E_0 ist also mit den F_λ^j vertauschbar und der Zerlegung $\mathbf{H} = \mathbf{H}_0 \oplus \mathbf{H}_1$, $\mathbf{H}_1 = (\mathbf{H}_0)^\perp$, entspricht eine Zerlegung in eine direkte Summe für P_ν und Q_ν:

$$P_\nu = P_{\nu 0} \oplus P_{\nu 1}, \quad Q_\nu = Q_{\nu 0} \oplus Q_{\nu 1}. \tag{22}$$

Dann gilt aber auch

$$P_\nu^2 = P_{\nu 0}^2 \oplus P_{\nu 1}^2, \quad Q_\nu^2 = Q_{\nu 0}^2 \oplus Q_{\nu 1}^2, \tag{24}$$

$$P_\nu^2 + Q_\nu^2 = (P_{\nu 0}^2 + Q_{\nu 0}^2) \oplus (P_{\nu 1}^2 + Q_{\nu 1}^2),$$

$$(P_\nu^2 + Q_\nu^2)^\frown = (P_{\nu 0}^2 + Q_{\nu 0}^2)^\frown \oplus (P_{\nu 1}^2 + Q_{\nu 1}^2)^\frown = A_\nu A_\nu^* - \frac{1}{2} I. \tag{25}$$

Es wird also auch der selbstadjungierte Operator

$$A_\nu A_\nu^* \quad \text{mit} \quad A_\nu = \frac{1}{\sqrt{2}} (Q_\nu + iP_\nu), \quad A_\nu^* = \frac{1}{\sqrt{2}} (Q_\nu - iP_\nu)$$

durch $\mathbf{H}_0$ reduziert. Also ist auch $A_\nu A_\nu^*$ mit P_μ, Q_μ vertauschbar und auch mit A_μ, A_μ^* und $A_\mu A_\mu^*$. Nun hat nach Satz 3 $A_\mu A_\mu^*$ ein rein diskretes Spektrum,

$$A_\mu A_\mu^* = \sum_0^\infty (m+1) E_\mu^m.$$

Da auch die Projektionen F_λ^j die Operatoren P_μ, Q_μ, damit auch $A_\mu A_\mu^*$ reduzieren, also mit den E_μ^m vertauschbar sind, folgt:[8])

Theorem 2. *Aus I, II, III folgt:*
Die Projektionen $\{E_\mu^m, F_\lambda^j\}$ *auf die Eigenräume der Operatoren* $A_\mu A_\mu^* = \sum_0^\infty (m+1) E_\mu^m$, $\Psi_\lambda \Psi_\lambda^* = F_\lambda^0 = I - F_\lambda^1$ *bilden ein kommutatives System.*
Die Operatoren $P_\nu, Q_\nu, A_\nu, A_\nu^*$ *werden durch alle* E_μ^m, F_λ^j $(\mu \neq \nu)$ *reduziert.*
Die Operatoren $\Psi_\lambda, \Psi_\lambda^*$ *werden durch alle* $E_\mu^m, F_\varkappa^j$, $\varkappa \neq \lambda$, *reduziert.*

Die Operatoren $A_\mu^* A_\mu$ haben die Eigenwerte $0, 1, 2, \ldots$. Diese Eigenwerte können also als *Besetzungszahlen des Zustandes* μ interpretiert werden. $\langle A_\mu^* A_\mu \varphi, \varphi\rangle = |A_\mu \varphi|^2$ wäre dann als *Erwartungswert für die Anzahl der Teilchen im Zustand* μ anzusprechen.

Entsprechend kann man die Eigenwerte 0, 1 von $\Psi_\lambda^* \Psi_\lambda$ als Besetzungszahlen für *Teilchen im Zustand* λ auffassen, wobei für diese Teilchen das Pauli-Prinzip gelten würde.

Ein Zustand φ_0 des Systems, bei dem alle möglichen Teilchenzustände unbesetzt sind, müßte also durch alle A_μ und Ψ_λ annulliert werden.

Wir verlangen nun weiter:

IV. In **H** gibt es einen „Vakuumszustand", d. h. ein φ_0, das durch alle A_μ und alle Ψ_λ auf Null abgebildet wird.

V. Das System $\mathfrak{S}$ ist irreduzibel in **H**.

Dann können wir zeigen, daß $\mathfrak{S}$ durch I—V bis auf unitäre Äquivalenz eindeutig festgelegt ist.

Theorem 3. *Das System* $\mathfrak{S} = \{P_\mu, Q_\mu, \Psi_\lambda, \Psi_\lambda^*\}$ *ist durch die Vertauschungsrelationen I—III, die Existenz eines Vakuumzustandes* φ_0 *(IV) und die Irreduzibilität (V) bis auf unitäre Äquivalenz festgelegt. Die eindeutige Lösung* $\mathfrak{S}$ *der Relationen I—V ist gegeben durch die Formeln* (28), (29), (30), (31).

Beweis. Die Menge $N = \{\lambda\}$ können wir mit einer Menge von natürlichen Zahlen identifizieren und erhalten dadurch eine vollständige Ordnung der Elemente λ. Der Vektor φ_0, der den Vakuumzustand repräsentiert, liegt in einem Eigenraum jedes der Operatoren $A_\mu A_\mu^* = A_\mu^* A_\mu + I$ und $\Psi_\lambda \Psi_\lambda^* = I - \Psi_\lambda^* \Psi_\lambda$. Auf φ_0 können wir also jeden der Operatoren Ψ_λ^* bzw. A_μ^* anwenden. Dabei wird nur der Eigenwert für $\Psi_\lambda \Psi_\lambda^*$ bzw. $A_\mu A_\mu^*$ geändert, $\Psi_\varkappa \Psi_\varkappa^*$ und $A_\nu A_\nu^*$, $\varkappa \neq \lambda$, $\nu \neq \mu$ haben für φ_0 wie für $\Psi_\lambda^* \varphi_0$ bzw. $A_\mu^* \varphi_0$ den gleichen Eigenwert. Wir können die Anwendung von Operatoren Ψ^* und A^* beliebig oft wiederholen, bleiben immer in der Menge der simultanen Eigenvektoren der Familie $\{A_\mu A_\mu^*, \Psi_\lambda \Psi_\lambda^*\}$. Es sei nun

$$(26)\quad \Delta_0 = \{(m, n);\, m = \{m_\mu\},\, n = \{n_\lambda\},\, m_\mu \in \{0, 1, 2, \ldots\},\, n_\lambda \in \{0, 1\},\, \textstyle\sum m_\mu < \infty,\, \sum n_\lambda < \infty\}.$$

Dann existiert für jedes $(m, n) \in \Delta_0$ der Vektor

$$(27)\qquad \varphi_{m,n} = \prod_\mu \frac{1}{\sqrt{m_\mu!}} (A_\mu^*)^{m_\mu} \prod_\lambda (\Psi_\lambda^*)^{n_\lambda} \varphi_0 .$$

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[8]) Theorem 1 und 2 zeigen, daß bei GÅRDING—WIGHTMAN [2], [3] statt der v. Neumann—Weylschen Form der Vertauschungsrelationen auch unsere Bedingungen I, II, III benutzt werden können.

Dabei sollen die Faktoren Ψ^*_λ nach wachsenden λ angeordnet sein. Es gelten wegen Satz 5 und Satz 6 die Relationen

$$A_{\mu_0}\varphi_{m,n} = \sqrt{m_{\mu_0}}\,\varphi_{m-\delta_{\mu\mu_0},n}, \tag{28}$$

$$A^*_{\mu_0}\varphi_{m,n} = \sqrt{m_{\mu_0}+1}\,\varphi_{m+\delta_{\mu\mu_0},n}, \tag{29}$$

$$\Psi_{\lambda_0}\varphi_{m,n} = (1)^{\sum\limits_{\lambda<\lambda_0} n_\lambda} \cdot n_{\lambda_0}\cdot\varphi_{m,n-\delta_{\lambda\lambda_0}}, \tag{30}$$

$$\Psi^*_{\mu_0}\varphi_{m,n} = (-1)^{\sum\limits_{\lambda<\lambda_0} n_\lambda} (1-n_{\lambda_0})\,\varphi_{m,n+\delta_{\lambda\lambda_0}}. \tag{31}$$

Dabei ist $\delta_{\mu\mu_0}$ bzw. $\delta_{\lambda\lambda_0}$ die Folge, die nur an der Stelle μ_0 bzw. λ_0 eine 1, sonst lauter Nullen hat. A_{μ_0} bzw. Ψ_{λ_0} verkleinern also den Index m_{μ_0} bzw. n_{λ_0} um 1, während $A^*_{\mu_0}$ bzw. $\Psi^*_{\lambda_0}$ diesen um 1 vergrößern. ($\varphi_{m,n}$ mit negativen m_μ und $n_\lambda \neq 0, 1$ sind dabei gleich Null zu setzen). Offenbar folgt nun

$$A^*_{\mu_0}A_{\mu_0}\varphi_{m,n} = m_{\mu_0}\varphi_{m,n}, \qquad \Psi^*_{\lambda_0}\Psi_{\lambda_0}\varphi_{m,n} = n_{\lambda_0}\varphi_{m,n}.$$

Wegen

$$\langle\varphi_{m,n},\varphi_{m,n}\rangle = \langle\varphi_{m-\delta_{\mu\mu_0},n},\varphi_{m-\delta_{\mu\mu_0},n}\rangle$$
$$= \langle\varphi_{m,n-\delta_{\lambda\lambda_0}},\varphi_{m,n-\delta_{\lambda\lambda_0}}\rangle = \langle\varphi_0,\varphi_0\rangle = 1$$

bilden also die $\varphi_{m,n}$, $(m,n)\in\Delta_0$, ein Orthonormalsystem in **H**. Der durch diese $\varphi_{m,n}$ aufgespannte Raum $\mathbf{H}_0$ ist invariant bei $\mathfrak{S}$, muß also nach (V) mit **H** übereinstimmen.

Die Operatoren A_μ, A^*_μ, Ψ_λ, Ψ^*_λ sind aber durch (28), (29), (30), (31) als abgeschlossene Operatoren eindeutig festgelegt (vgl. Satz 5) und nach Theorem 1 sind dann auch $P_\mu = \frac{i}{\sqrt{2}}(A^*_\mu - A_\mu)^{\sim}$ und $Q_\mu = \frac{i}{\sqrt{2}}(A^*_\mu + A_\mu)^{\sim}$ durch (28) und (29) vollständig bestimmt.

Damit ist aber dann die Eindeutigkeit von $\mathfrak{S}$ bis auf simultane unitäre Transformationen nachgewiesen.

Ist $\mathfrak{S} = \{P_\mu, Q_\mu, \Psi_\lambda, \Psi^*_\lambda\}_{\mu\in M,\lambda\in N}$ ein endliches System, also M und N endliche Indexmengen, so ist $\{A^*_\mu A_\mu, \Psi^*_\lambda\Psi_\lambda\}_{\mu\in M,\lambda\in N}$ ein System von endlich vielen, paarweise vertauschbaren selbstadjungierten Operatoren, welche alle diskretes Spektrum und 0 als Eigenwert haben. Es gibt also dann mindestens einen gemeinsamen Eigenvektor zum Eigenwert 0, d. h. (IV) ist eine Folge von (I, II, III).

Es sei nun $\mathbf{H}_0$ der Raum aller gemeinsamen Eigenvektoren der $A^*_\mu A_\mu$ und $\Psi^*_\lambda\Psi_\lambda$ zum Eigenwert 0, $\varphi_0^{(\alpha)}$ eine Orthonormalbasis von $\mathbf{H}_0$.

Dann können offenbar die Vektoren

$$\varphi^{(\alpha)}_{m,n} := \prod_\mu \frac{1}{\sqrt{m_\mu!}}(A^*_\mu)^{m_\mu}\prod_\lambda(\Psi^*_\lambda)^{n_\lambda}\varphi_0^{(\alpha)}$$

gebildet werden. Der durch die $\varphi^{(\alpha)}_{m,n}$ mit festem α aufgespannte Raum $\mathbf{H}^{(\alpha)}$ reduziert

$A_\lambda, A_\mu^*, P_\mu, Q_\mu, \Psi_\lambda, \Psi_\lambda^*$ und es gilt:

$$A_\nu \varphi^{(\alpha)}_{m,n} = \sqrt{m_\mu}\, \varphi^{(\alpha)}_{m-\delta_{\mu\nu},\, n}, \quad A_\nu^* \varphi^{(\alpha)}_{m,n} = \sqrt{m_\nu + 1}\, \varphi^{(\alpha)}_{m+\delta_{\mu\nu},\, n},$$

$$\Psi_\varkappa \varphi^{(\alpha)}_{m,n} = (-1)^{\sum_{\lambda<\varkappa} n_\lambda} \cdot n_\varkappa \varphi^{(\alpha)}_{m,\, n-\delta_{\lambda\varkappa}}, \quad \Psi_\varkappa^* \varphi^{(\alpha)}_{m,n} = (-1)^{\sum_{\lambda<\varkappa} n_\lambda} (1-n_\varkappa) \varphi^{(\alpha)}_{m,\, n+\delta_{\lambda\varkappa}}.$$

Hieraus folgt zusammen mit Satz 5 und Theorem 1:

Theorem 4. *Ist* $\mathfrak{S} = \{P_\mu, Q_\mu, \Psi_\lambda, \Psi_\lambda^*\}_{\mu\in M,\, \lambda\in N}$ *mit endlichen Indexmengen* M, N *und sind die Bedingungen (I), (II), (III) erfüllt, so gilt:*

Es gibt eine Zerlegung $\mathbf{H} = \oplus \mathbf{H}^{(\alpha)}$ *von* $\mathbf{H}$ *in paarweise orthogonale Teilräume* $\mathbf{H}^{(\alpha)}$, *welche alle Operatoren aus* $\mathfrak{S}$ *reduzieren. Die irreduziblen Systeme* $\mathfrak{S}^{(\alpha)} = \{P_\mu^{(\alpha)}, Q_\mu^{(\alpha)}, \Psi_\lambda^{(\alpha)}, \Psi_\lambda^{(\alpha)*}\}$ *in* $\mathbf{H}^{(\alpha)}$ *sind bis auf unitäre Äquivalenz bestimmt.*

Insbesondere sind also quantenmechanische Systeme von endlich vielen Freiheitsgraden vollständig bestimmt durch endlich viele Paare P_μ, Q_μ, für die II gilt und P_ν, Q_ν mit P_μ, Q_μ für $\nu \neq \mu$ vertauschbar sind, und durch den Hamiltonoperator $H = H(P_\mu, Q_\mu)$. Die irreduziblen Bestandteile der P_μ, Q_μ sind den Schrödingerschen Operatoren äquivalent:

$$P_\mu \varphi = \frac{1}{i}\frac{\partial}{\partial x_\mu}\varphi; \qquad D_{P_\mu} = \left\{\varphi;\ \varphi \in L^2(R^M),\ \varphi \text{ absolut stetig, } \frac{\partial \varphi}{\partial x_\mu} \in L^2\right\},$$

$$Q_\mu \varphi = x_\mu \cdot \varphi; \quad D_{Q_\mu} = \{\varphi;\ \varphi \in L^2(R^M),\ x_\mu \cdot \varphi \in L^2(R^M)\}$$

und die Schrödingergleichung $-\frac{1}{i}\frac{\partial}{\partial t}\varphi = H\varphi$ kann in der gewohnten Weise als partielle Differentialgleichung geschrieben werden:

$$\frac{1}{i}\frac{\partial}{\partial t}\varphi + H\left(\frac{1}{i}\frac{\partial}{\partial x_\mu}, x_\mu\right)\varphi = 0.$$

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Bibliographie

W. Rinow, Die innere Geometrie der metrischen Räume (Die Grundlehren der mathematischen Wissenschaften, Band 105), XV+520 Seiten, Berlin—Göttingen—Heidelberg, Springer-Verlag, 1961.

Das Buch gibt einen systematischen Aufbau der inneren Geometrie der metrischen Räume. In hohem Maße wurden topologische Hilfsmittel verwandt, die ermöglichten, daß viele klassische Begriffe bzw. Resultate der Differentialgeometrie ohne Differenzierbarkeitsvoraussetzungen formuliert werden konnten.

Die behandelten Problemkreise von dem Gebiet der inneren Geometrie sind in zehn Kapiteln zerteilt. Im ersten Kapitel sind die Grundbegriffe der metrischen Geometrie und die im folgenden nötigen Grundideen der Topologie angegeben. Das zweite Kapitel behandelt die stetigen Abbildungen und entwickelt den Begriff der Kurven vom Standpunkt der Topologie und auch als parametrisierte Punktmengen.

Nach diesen einleitenden Betrachtungen wird im dritten Kapitel die Länge einer Kurve und die innere Metrik eines metrischen Raumes definiert. Als Beispiel der inneren Metriken wird die Metrik der Finslerräume eingehender behandelt.

In den folgenden drei Kapiteln folgt eine ausführliche Behandlung der Theorie der geodätischen Kurven.

Im siebenten Kapitel ist der in der Theorie der metrischen Räume so wichtige Begriff der Krümmung definiert. Dieser Begriff beruht auf dem Vergleich der Dreiecke des Raumes mit kongruenten Dreiecken auf Flächen konstanter Krümmung. Die Bestimmung der Krümmung ist auch durch den Dreiecksexzeß angegeben, die bezüglich der vorigen Definition einen einfacheren Charakter hat.

Das achte Kapitel behandelt das Clifford—Kleinsche Raumformenproblem. Das Clifford—Kleinsche Raumformenproblem bedeutet die Aufgabe der Bestimmung aller Mannigfaltigkeiten konstanter Riemannscher Krümmung. Die Bestimmung der euklidischen, sphärischen und hyperbolischen Raumformen geschieht nach einen von F. Klein stammenden Lösungsverfahren mittels der Decktransformationsgruppen des Basisraumes.

Das neunte Kapitel enthält Untersuchungen über Räume, deren Krümmung $\leqq 0$ ist. Ein Raum mit der Krümmung <0 *ist im wesentlichen durch folgende Forderung definiert:* Bilden die Punkte a, b, c, ein nichtausgeartetes Dreieck, sind ferner K_{ab} bzw. K_{ac} die Kürzesten zwischen a, b bzw. a, c und sind x, y die Mittelpunkte von K_{ab} bzw. K_{ac}, so gilt $\varrho(x, y) < \frac{1}{2}\varrho(b, c)$, wo die Funktion $\varrho(x, y)$ den Abstand der Punkte x, y bedeutet. Viele interessante Einzelprobleme werden diskutiert.

Im letzten Kapitel untersucht der Verf. die Sphäroide und Räume vom elliptischen Typ.

Das interessante Buch von Prof. Rinow wird gewiß zu weiteren Untersuchungen Anlaß geben.

A. Moór (Szeged)

A. H. Clifford and G. B. Preston, The Algebraic Theory of Semigroups, Vol. I (Mathematical Surveys, Number 7), 224 pages, Providence, American Mathematical Society, 1961.

In the afterwar period there took place a rapid development in the theory of semigroups, i. e. algebraic systems with an associative binary operation. In the last decade the theory of semigroups has become an independent branch of mathematics, which has its own aspect, its own methods and contains a great deal of valuable results, sometimes rather deep and apt for application. Some results and methods of the semigroup theory are applied in functional analysis, in the general theory of transformations, and also in the abstract theory of automata, still being in statu nascendi.

Mathematicians interested in semigroups may be satisfied by the fact that in the last few years several books have appeared dealing with the systematic study of these objects, such as the

"Semigroups" by E. S. LYAPIN (Moscow, 1960, in Russian), the present book of A. H. CLIFFORD and G. B. PRESTON, and the quite recent book "Die Theorie der endlich erzeugbaren kommutativen Halbgruppen" (Leipzig and Hamburg, 1962), by L. RÉDEI.

The book of CLIFFORD and PRESTON contains a suitably selected rich material representing the most interesting and important investigations in semigroups. It is written in a clear and easily comprehensible style.

Let us give a short review of the contents of the book, which consists of 5 chapters and an Appendix, containing some results taken from the book of A. K. SUSHKEWITCH "The Theory of Generalized Groups" (Kharkow, 1937, in Russian). In the first chapter the basic concepts and definitions are given. The associativity test of F. W. LIGHT for finite gruppoids is presented. In § 3 some results concerning translations and (regular) representations are included. The further §§ deal with the semigroup of all binary relations on a set; the concepts of congruence, factor semigroup, homomorphism, semilattice and band of semigroups are introduced and some general theorems proved. The class of inverse semigroups is treated separately and the remark is made that in volume 2 the authors will give a more detailed study of this important class of semigroups. § 10 deals with the problem of embedding of semigroups into groups. At the end of the chapter the concepts of the free semigroup and generating relations are introduced.

In chapter 2 the ideals of semigroups are considered. § 1 introduces the concept of GREEN's equivalence classes. Further, some special cases are treated, such as the $\mathfrak{D}$-structure of the transformation semigroups of sets and the regular $\mathfrak{D}$-classes of arbitrary semigroups. There are to be found some investigations on semigroups with zero element, containing minimal ideal and proofs theorems analogical with the well-known JORDAN—HÖLDER—SCHREIER theorems in group theory. The chapter ends by some results preparing the study of completely 0-simple semigroups.

Chapter 3 deals with the representations of semigroups by means of matrices the elements of which belong to a group with zero element, and the famous theorem of REES is discussed, enabling us to give a complete description of completely 0-simple semigroups. The authors further present the application of this theorem to Brand gruppoids and then give a description of all the homomorphisms of completely 0-simple semigroups. The last two §§ treat special (Schützenberger) representations of semigroups and the true representations of regular semigroups.

Chapter 4 is devoted to the theory of the decompositions of semigroups into the union of semigroups of various kinds (including groups) and the ideal extensions of semigroups.

Chapter 5 treats the representations of semigroups by means of matrices over a field, the principal irreducible representations of a semigroup, as well as representations of completely 0-simple semigroups.

The book ends by reviewing some important results of the theory of the characters of commutative semigroups due to Š. SCHWARZ, E. HEWITT and H. S. ZUCKERMAN.

In the book there are also some very valuable exercises to be solved and some open questions mentioned.

We are looking forward with much interest to the second part of the work.

I. Peák (Szeged)

H. S. M. Coxeter, Unvergängliche Geometrie. Ins Deutsche übersetzt von J. J. BURCKHARDT, 552 Seiten, Basel und Stuttgart, Birkhäuser Verlag, 1963.

Den viel hundert- sogar viel tausendjährigen Stoff der Geometrie vermag man immer wieder in neuer Gruppierung vorzutragen. Dennoch ragt COXETERS Buch aus der Literatur ähnlichen Charackters hervor und ist vom großen Nutzen für die Forscher und Lehrer der Geometrie. Der stets im Vordergrund stehende Leitfaden des Werkes ist: aus jeder mathematischen Konstruktion den geometrischen Kern hervorzuheben und den Stoff der Geometrie mit dem Felix Klein-schen Gedanken zu vereinigen.

Die Lehrer der Geometrie finden in diesem Buch schöne und lehrreiche Anwendungen, undzwar nicht nur die gebräuchlichen kosmologischen, kinematischen, sondern auch die weniger bekannten kristallographischen und botanischen Anwendungen. Die Thematik ist sehr umfangreich: die nicht-Euklidische Geometrie, Differentialgeometrie, elementare Topologie (in selbstständigem Aufbau, mit dem Vierfarbenproblem im Mittelpunkt) bekommen ihren gehörigen Platz. Neuartig sind u. a. die Benützung von Dominos zur Illustration von sechs unter den siebzehn Bewegungsgruppen der ebenen Kristallographie (§ 4.4), ein sparsam gewähltes Axiomensystem der affinen Geometrie (§ 15.4), eine elementare Behandlung der extremen quadratischen Formen (18.4), eine Anwendung von geodätischen Polarkoordinaten auf die Begründung der hyperbolischen Trigonometrie (§ 20.6) und die Behandlung der geometrischen Transformationen (§ 5.6, 6.7, 7.6, 15.4).

Die Lektüre des Buches machen die Zitaten am Anfang der einzelnen Kapitel sehr anziehend. Dem Text sind 500 Aufgaben beigefügt; die Lösungen sind am Ende des Buches kurz angedeutet. Die bibliographischen Hinweise sind zeitgemäß und sehr reichlich. Einige Kapitel des Buches sind auch für die Weiterbildung von Mittelschullehrern zu empfehlen, so z. B. gleich das erste Kapitel über die Dreicke.

J. Berkes (Szeged)

Konrad Jacobs, Neuere Methoden und Ergebnisse der Ergodentheorie (Ergebnisse der Mathematik und ihre Grenzgebiete, neue Folge, Heft 29), IV+214 Seiten, Berlin—Göttingen—Heidelberg, Springer-Verlag, 1960.

Although the development of ergodic theory has taken place principally only since BIRKHOFF's paper in 1931, there exists already a huge literature on this subject. Since E. HOPF's excellent monograph (1937), treating the early development of the theory, there appeared no other monograph on the subject which would give a comprehensive account on the whole theory including its recent results and methods. The present work fills this gap, by giving a rather complete account of the pertaining literature up to 1958.

In contains nine chapters and ends with a very complete bibliography. For the convenience of the reader, the notions and facts of functional analysis and measure theory, which are used in the book, are summarized in the last two chapters (8 and 9).

Chapter 1 is devoted to functional analytic ergodic theorems, the fundamental problem of which is formulated as a fixed point-problem. First the ergodic theorem of ALAOGLU and BIRKHOFF is presented together with its genaralizations and related topics (§§ 1—2), then there follow general functional analytic recurrence theorems (§ 6). § 7 contains the author's results concerning the relation between the reversibility and almost periodicity of vectors of a Banach space. After giving a necessary and sufficient condition in order that a vector of a Hilbert space be almost periodic with respect to a semi-group of contractions (§ 8), the chapter ends with norm-convergence theorems of martingales (§ 9).

The results of Chapter 1 are applied in Chapter 2 to stationary Markov processes.

Chapter 3 treats the individual ergodic theorem in various forms. In § 1 the individual ergodic theorem is proved for discrete pointflows. Next this theorem is proved for discrete and continuous operator-flows (§ 2—3). § 4 deals with the case of non-singular measurable transformations, which are not necessarily measure-preserving. Generalizations and related questions are treated in § 5. At the end of this chapter, the a. e. convergence of discrete martingales is discussed.

In Chapter 4 the concepts of recurrency and ergodicity and those of strong and weak mixing are introduced (§ 1—4) with some important examples for these notions (§ 5). § 6 contains theorems concerning the decomposition into ergodic parts of measure spaces. § 7 discusses normal forms of measure-preserving flows, and § 8 treats the existence problem of invariant measures.

In Chapter 5 flows are studied which consist of continuous mappings of a topological space Ω into itself (topological flows). To begin with, in § 1 some results on these flows are presented, which can be proved by pure topological methods. More far-reaching results can be obtained when the measure theory of the Borel structure deduced from the topology of Ω is used. The corresponding theorems are given in § 2.

Chapter 6 is devoted essentially to the study of various topologies of the group G of all invertible measure-preserving transformations of the interval $\langle 0, 1)$, the latter being considered as a measure space with respect to Lebesgue measure. After introducing the notions of periodicity and antiperiodicity for elements of G, the notion of permutations, the strong (metric) and weak topologies of G, the author presents important density theorems (§§ 1—5). In § 6, cathegory theorem for the ergodic, weak, and strong mixing elements of G are given.

Chapter 7 deals with non stationary problems. In § 1 random ergodic theorem are proved, and § 2 is devoted to non stationary Markov processes.

In spite of its relatively short extent, the book is well-readable and the most important results of ergodic theory are presented in it with complete proofs.

I. Kovács (Szeged)

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LADISLAUS RÉDEI

THEORIE DER ENDLICH ERZEUGBAREN KOMMUTATIVEN HALBGRUPPEN

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Nicht nur in der Algebra, sondern in allen Zweigen der Mathematik hat die Theorie der Halbgruppen eine große Bedeutung. Ihre Untersuchung erfuhr jedoch erst in den letzten drei Jehrzehnten einen lebhaften Aufschwung, und in den jüngsten Jahren erschienen die ersten zwei diesbezüglichen Monographien. Es fehlte aber bisher noch der Ausbau einer Theorie der endlich erzeugbaren kommutativen Halbgruppen, die eine verhältnismäßig einfache und deshalb sehr wichtige Klasse von Halbgruppen bilden. Diesem Mangel abzuhelfen, ist das Ziel der vorliegenden Monographie. Die darin ausgebaute Theorie ist von vollständiger grundsätzlicher Einfachheit. Sie beruht auf einem einzigen Fundamentalsatz, der grob gesprochen aussagt, daß die endlich erzeugbaren kommutativen Halbgruppen sich durch sogenannte (axiomatisch definierte) Kernfunktionen beschreiben lassen, so daß es sich dann im Grunde um die Theorie der Kernfunktionen handelt. Die Anwendbarkeit der Theorie wird unter anderem durch mehrere „Endlichkeitssätze" gesichert. Ein Teil dieser Sätze enthält leicht erfaßbare, der andere Teil tiefliegende Feststellungen. Die wichtigsten zwei unter ihnen sprechen aus, daß jede endlich erzeugbare kommutative Halbgruppe auch endlich (d. h. durch endlich viele Gleichungen) definierbar ist und jede Kernfunktion einen endlichen Wertevorrat hat. Die Betrachtungen sind rein algebraisch, sind jedoch auf natürliche Weise in ein geometrisches Gewand gekleidet, wodurch die Ausführungen beträchtlich erleichtert werden und die Theorie als ein Kapitel der (mehrdimensionalen) Gitterpunktsgeometrie erscheint. Der Verfasser arbeitet zahlreiche Beispiele ausführlich aus, die den Leser zum Weiterforschen anregen und befähigen. Für Algebraiker, Mathematiker, Dozenten und Studenten an Universitäten wird das Studium dieser Theorie von hohem Interesse sein.

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